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"Nec aranearum sane textus ideo melior quia ex se fila gignunt, nec noster vilior quia ex alienis libamus ut apes." Just. Lips. *Polit.* lib. i. cap. 1. Not.

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[FIFTH SERIES.]

JUNE 1876.

LI. *On the Simultaneous Sounding of two Notes.* *By* Dr. RUDOLPH KÖNIG, *Paris**.

IF two notes are produced upon the same instrument, or by the vibrations of two bodies which are closely connected together through a third, there ensue some very intricate phenomena, which are partly produced by the reaction of the two sources of sound upon each other, and the action of both upon the connecting body, and partly also have their origin in the continuance of the two sound-waves in the air. It is my present intention in the following pages to submit to a closer examination only those phenomena which arise from the coexistence of two sound-waves in the air; and I have therefore used for the demonstration of these waves only such sources of sound as were absolutely isolated from each other, and could not possibly act upon each other directly, or combined together upon a third body. As, further, the waves produced by clangs† must always be considered as a combination of waves of simple notes, and as therefore it may remain doubtful when clangs are employed whether the phenomena observed are produced by the fundamental notes or by the over-notes, I have been careful in these experiments so to select the sources of sound that they should only produce the simplest possible notes. For the low notes I used very stout tuning-forks, mounted on isolated iron frames, and placed in front of large sounding-boxes; for the upper notes

* Translated from Poggendorff's *Annalen*, vol. clvii. p. 177. And communicated by W. Spottiswoode, M.A., LL.D., V.P.R.S.

† By "clang" is meant the entire sound emitted by an instrument when sounding a musical note.

simply powerful tuning-forks, whose intensity of tone required no further reinforcement.

The whole series of tuning-forks and sounding-boxes which I made use of for these experiments was as follows:—

1. Five tuning-forks which without weights gave the notes double G, C, E, G, *c* (sol_{-1}, do_1, mi_1, sol_1, do_2). Each of the four higher forks can, by means of its sliding weights, be tuned down to the note of the next deeper fork. The double-G fork can be lowered by a couple of sliding weights to double E, and by another couple as far as to double C ($do_{-1}=64$ v. s.); and this latter limit may still be exceeded by increasing the weights upon the slides. The positions of the sliding weights upon these forks are marked at intervals of a single vibration for the octave from double C to C, and by a double vibration for the next higher octave.

The prongs of the lowest fork are 35 millims. in thickness, 55 millims. in breadth, and about 75 centims. in length. The prongs of the other four forks are 39 millims. thick, 55 millims. broad, and their length varies from about 70 centims. to 49 centims. These five forks, without their stands and sliding weights, weigh 130 kilogrammes.

2. Eight tuning-forks, which without weights are tuned to the notes $c, e, g, c', c', e', g', c''$ (do_2, mi_2, sol_2, do_3, do_3, mi_3, sol_3, do_4), and by means of their sliding weights are capable of producing all the intermediate notes also. Their prongs are 26 millims. thick, 26 millims. broad, and from about 59 to 19 centims. long.

The tuning-forks for the octave from c to c' have a division for placing the sliding weights at from two to two double vibrations, and for the next higher octave from four to four.

3. Nine forks tuned to the scale of c'' to c''', and to the seventh harmonic of the small c, of which the prongs are 25 millims. broad, 25 millims. thick below, and diminish to about 12 millims. less at the ends. Their length varies from about 20 to 13 centims.

4. Twelve tuning-forks for the notes of the scale from c''' to c^{IV}, the eleventh, thirteenth, and fourteenth harmonics of c, and the note of 2389·3 v. s., which forms with c' (512 v. s.) the ratio of 3 : 14, with prongs 15 millims. in breadth, 10 millims. in thickness below, and about 7 millims. at the ends, and from about 9 centims. to 6 centims. in length.

5. Eleven forks for the notes of the scale from c^{IV} to c^{V}, and the eleventh, thirteenth, and fourteenth harmonics of c'. The breadth of the prongs is 23 millims., the thickness below 18 millims., and at the ends about 9 millims. Their length varies from about 8 to 5 centims.

6. A series of eleven tuning-forks for notes between b''' and c^{v}, and a series of nine tuning-forks for notes between 7936 v. s. and c^{v} (8192 v. s.), with prongs 14 millims. in breadth, and below about 8 millims. in thickness.

7. Three pairs of resonators to strengthen the notes from C to c'', provided with clamping-screws, so that they may with the greater precision be tuned to each note which requires reinforcement. They are made of brass, and mounted on iron frames. At the opening of each of these resonators two side plates can be applied, in case the tuning-forks cannot, in consequence of their weights, be brought near enough to the opening, and it is desirable to lose as little as possible of their action on the body of air. Besides this, each clamp, close to the screw which goes through and moves it, is bored through, and provided with a small pipe, which is generally closed, but which can be opened, in order that by its means, with the help of an india-rubber tube, the ear may be put in direct communication with the body of air within the resonator. The two resonators which reinforce the notes from C to g are 30 centims. in diameter, 1 metre 15 centims. in length, and the opening in front is 27 centims. long and 12 centims. broad. The two resonators which can be tuned from C to g' are 25 centims. in diameter, 50 centims. in length, and their opening is 23 centims. in length and 7 centims. in breadth. The third pair of resonators reinforce the notes from g to c''. Their length is 36 centims., their diameter 25 centims., and their opening is 15 centims. in length and 7 centims. in breadth.

I. *Primary Beats and Beat-Notes.*

A. *Intervals with the fundamental note* C (=128 *v. s.*).

If at the same time with the deep, simple, and strong note C (128 v. s.), produced by means of a large tuning-fork placed in front of a resonator, a second note is produced in the same manner, which, starting from unison, is gradually raised higher and higher, the beats which ensue after the disturbance of the unison become gradually quicker. When the higher note has reached to 152 or 156 v. s. (that is, between D and E), the beats, which till then were heard separately to the number of twelve to fourteen, change to a roll, which increases in rapidity till the interval of the fourth is nearly reached, about 171 v. s. (twenty-two beats), without losing its simple character. When the fourth is passed, there occurs a confused but always very loud rattle, which lasts till above the fifth, until when close upon the sixth, at about 212 to 216 v. s., it begins to lose its confused character, and changes into a still rapid but simple roll, which becomes so much slower between the sixth and the seventh that at 233 and 236 v. s. twelve and ten single beats can already be counted, which at the seventh, $B=240$ v. s., decrease to eight, at 244 v. s. to six, and become gradually fewer in number, till at the octave of $c=256$ v. s. they at last cease altogether.

As the number of vibrations of the primary notes can be directly read off from the tuning-forks, it will be found that the number of single distinguishable beats near the unison is equal to the difference of the double vibrations of the two primary notes, and that of the beats near the octave is equal to the difference of the double vibrations of the higher of the two primary and of the octave of the lower note.

The above result can be shortly expressed in the following manner. Each interval $n : n'$ (less than the octave) exhibits two sorts of beats, whose number is equal to the positive and negative remainder of the division $\frac{n'}{n}$, that is to say, equal to the two numbers m and $m'=n-m$, which we obtain by stating $n'=n+m=2n-m'$. I shall in future, for the sake of brevity, call the beats m lower beats, and the beats m' upper beats. If we increase the interval between two notes from the unison to the octave, the number of lower beats increases from o to n, and that of the upper beats diminishes from n to o. At the fifth the number of both kinds of beats is $=\frac{n}{2}$. If m is much less than $\frac{n}{2}$, only the lower beats are audible; if m is much greater than $\frac{n}{2}$, we hear only the upper beats; and if m is nearly equal to $\frac{n}{2}$, both kinds of beats, m and $n-m$, may be distinguished at the same time.

The lower beats are more powerful than the upper beats; and their audibleness extends consequently further beyond the fifth than that of the upper beats below the same note.

In the octave from C to c, which we have now been considering, it is very difficult to distinguish, through the loud and confused rattle of the upper and lower beats above and below the fifth, the rhythm which belongs to both these kinds of beats, as the number both of lower and upper beats is always so great that heard alone they would produce a very rapid roll. I only succeeded, therefore, in bringing to special and absolutely clear proof both kinds of beats during their existence by choosing as the fundamental note of the interval a much deeper note than C, namely double E (80 v. s.).

The great fork had on one of its prongs a wooden board 24 centims. broad and 40 centims. long, and, by means of a powerful electromagnet placed between its prongs, was made to vibrate strongly in a space of from 12 to 15 millims. I held my ear to this board, while I brought more or less close to it a tuning-fork with sliding weights and divisions, which I held loose in my hand. If we experiment in this manner, and raise the note on the latter tuning-fork from 80 v. s. gradually higher and higher, the first single audible beats are again lost in a roll and rattle which continue beyond the fifth (20 beats). At 144 v. s., when 32 lower beats and 8 upper beats are produced, the latter begin to be distinguishable. At 148 v. s. ($m=34$, $m'=6$) and at 150 v. s. ($m=35$, $m'=5$), beside the rattle of the 34 or 35 lower beats, the 6 and 5 upper beats can be clearly heard. A good idea may be obtained of the sound produced if we curve the tongue as in forming the letter R, while we force the air out of the mouth in short strong puffs, instead of in one continuous stream.

While on the subject of this experiment with the deep double-E fork, I may remark by the way that it is extremely difficult to produce very deep simple notes of any intensity. As I was anxious to make my experiments upon beats by means of notes which should have the widest possible intervals with the smallest possible difference in the absolute number of vibrations, I constructed for the notes of the double octave (64–188 v. s.) two large wooden resonators, one 40 and the other 60 centims. high and wide, and both 2 metres long. Like the before-mentioned brass resonators, they were fitted with clamping-screws, so that they could be tuned with the greatest precision, and the openings could be increased or diminished at pleasure; but the effect which I obtained from them in connexion with the powerful forks was so small that I lost more in intensity by taking one of these deep notes for the fundamental note than I gained by the lesser number of vibrations.

If we increase the interval of the octave 128 : 256 v. s., at which we have now arrived by retaining the fundamental note C, while we again raise the second note proceeding from the octave higher and higher, we produce again the single audible beats, which, when they have attained the number of 10 to 12 at 276 to 280 v. s., change into a simple roll, which at 296 v. s. (20 beats) is transformed into a confused rattle. This rattle soon becomes weaker; and between *e* and *f*, about 332 to 336 v. s., the clang of the two notes only allows a mere roughness to be perceived, out of which, however, again at 344 v. s., a clear quick roll appears, which gradually becomes slower, till at 360 to 364 v. s. 12 to 10 beats become singly audible; these at 368, 372, 376, and 380 v. s. decrease to 8, 6, 4, and 2, and at $g=384$ v. s. (1 : 3) disappear.

The number of the beats audible close to the octave is equal to the difference of the double vibrations of the higher notes and of the octave of the fundamental note; and the number of beats close to the twelfth is equal to the difference of the double vibrations of the higher notes and the twelfth from the fundamental note.

The order of the intervals here observed of this second period, from $n : 2n$ to $n : 3n$, is therefore precisely the same as that which we observed of the intervals of the first period from $n : n$ to $n : 2n$. Each interval, $n : 2n+m$ or $3n-m'$, again exhibits two kinds of beats, which are equal to m and m': if m is much less than $\frac{n}{2}$, we hear only the lower beats: if m is much greater than $\frac{n}{2}$, only the upper beats can be distinguished; and if m is about the same as $\frac{n}{2}$, the two kinds of beats exist together. In this period $m=\frac{n}{2}$ in the interval $2 : 5$ ($e=320$ v. s.).

The beats in the interval $n : 2n+m$ are therefore equal to those in the interval $n : n+m$.

In this period also the upper beats are weaker than the lower; and both upper and lower beats are weaker than the corresponding beats in the first period.

The next higher period reaches from $C : g$ to $C : c'$, $n : 3n$ to $n : 4n$; and its centre, in which $m=\frac{n}{2}$ is in the ratio of $2 : 7$ ($128 : 448$ v. s.).

We find in this the same order as in the two first periods, only we cannot follow the two kinds of beats quite so far, as they have again become weaker than in the former periods. If, beginning at g (384 v. s.), we again raise the second note higher and higher, the first single audible beats fall into a roll at 404 v. s. (10 beats), which at 420 v. s. becomes a weak confused rattle. This changes at about 456 v. s. to a mere roughness, from which another clear rattle is only distinguished at 480 to 484 v. s. (16 to 14 beats), which becomes slower till at 492 v. s. we have 10 single audible beats, gradually diminishing in number till at c' (512 v. s.), the double octave, they entirely disappear.

The beats of an interval $n : 3\,n + m$ or $4\,n - m'$ are again equal to m and m'.

In the period of $C : c'$ to $C : e'$, of $n : 4\,n$ to $n : 5\,n$, the beats can be followed to only a less distance. The lower beats at the number of 8 to 10 change into a roll; but this at 552 v. s. (20 beats) is already so weak that it is hardly more than a mere roughness. At 560 (24 beats) even this is no longer distinguishable, and the two notes from this point form a pure clang. Only at 616 v. s. does the roll of 12 beats again appear from the pure clang, which then passes into the single audible beats, which disappear at 1 : 5 (128 : 640 v. s.).

In the period $C : e'$ to $C : g'$, $n : 5\,n$ to $n : 6\,n$, the lower beats are only clear to about 10, and disappear at 664 v. s. (12 beats). The upper beats are feebly audible at 748 v. s., and only at 752 v. s. (8 beats) become singly quite clear.

In the period $C : g'$ to C : 896 v. s., of $n : 6\,n$ to $n : 7\,n$, the lower beats are only quite distinct up to 780 v. s. (6 beats), and disappear at 784 v. s. The upper beats are feebly audible to the number of 6 at 884 v. s., and only become quite distinct at 888 v. s. to the number of 4.

In the period of C : 896 v. s. to $C : c''$, $n : 7\,n$ to $n : 8\,n$, the lower beats can be heard clearly to the number of 4 at 904 v. s. They disappear at 908 v. s., 6 in number. Four upper beats are perceptible at 1004 v. s. The two beats at 1008 v. s. are quite distinct.

I succeeded occasionally in perceiving a few beats in the ratio of $C : d''$ and even of $C : e''$ (1 : 9 and 1 : 10); but these were very weak, and could not have been perceived at all by any ordinarily correct ear not specially trained for the purpose, as all those above described can be.

It has always hitherto been assumed that beats can only be directly produced from two notes which are close to the unison, and that the beats of all wider intervals must be produced with the aid of resultant notes. According to this, in the interval $C : c'' - 2$ v. d., which as we have seen allows two beats to be distinctly heard, these beats must have been produced in the following manner:—

$c''-2$ v.d.	with	C ($8\,n-2$ with n)	must have	produced	892 v. s. ($7\,n-2$)
892 v. s.	„	C ($7\,n-2$ „ n)	„	„	$g'-2$ v.d. ($6\,n-2$)
$g'-2$ v.d.	„	C ($6\,n-2$ „ n)	„	„	$e'-2$ v.d. ($5\,n-2$)
$e'-2$ v.d.	„	C ($5\,n-2$ „ n)	„	„	$c'-2$ v.d. ($4\,n-2$)
$c'-2$ v.d.	„	C ($4\,n-2$ „ n)	„	„	$g-2$ v.d. ($3\,n-2$)
$g-2$ v.d.	„	C ($3\,n-2$ „ n)	„	„	$c-2$ v.d. ($2\,n-2$)
$c-2$ v.d.	„	C ($2\,n-2$ „ n)	„	„	C -2 v.d. ($n-2$)
C -2 v.d.	„	C ($n-2$ „ n)	„	„	two beats

Of all these intermediate notes I have been able to discover no trace; and the note $c''-2$ v. d. (1020 v. s.) has besides comparatively so little intensity, even when its beats are most distinctly audible with C, that it seems absolutely impossible that it should produce any (even the very smallest) practical combination-note in connexion with other notes; and it would be still more incredible that it should be the origin of a whole series of combination-notes. It is therefore far more natural to derive the beats of the harmonic interval, as well as those of the unison, directly from the formation of the sound-waves, and to consider that they arise from the periodically interchanging coincidences of the similar maxima of the notes n and n', and of the maxima which have opposite signs.

In the beats of these harmonic intervals, as well as in those of the unison, the similar maxima will either come exactly together, or else with two successive vibrations of the fundamental note; maxima of compression of the higher notes will slightly precede the maximum of compression of the first vibration and follow the second, so that the centre of beating will lie between these two; in both cases, however, the effect upon the ear will be exactly the same, as a beating is no momentary phenomenon, but arises from the constant ebb and flow of the intensity of the note. To give a clearer idea of the order of vibrations in the beats of these harmonic intervals, I have reduced to writing the vibrations of the interval $n:hn$ and $n:hn+y$ $(h=1, 2, \ldots 8)$ by means of my well-known apparatus, with which, according to the method first applied by Lissajous and Desains, one of the tuning-forks whose vibrations are to be calculated has attached to it a piece of smoked glass which vibrates with it, and the other carries the pencil which marks the figures upon this plate. If we look at the common characters of these figures, we find that the beats of the imperfect intervals 1 : 3, 1 : 5, 1 : 7, as well as the beats of the unison, are shown by periodical maxima and minima of the amplitude of the vibrations which very clearly declare their direct audibleness. In the perfect intervals, 1 : 2, 1 : 4, 1 : 6, and 1 : 8, a maximum of compression is constantly changing with a maximum of dilatation, as is the case in ordinary sound-waves, and every entire period may therefore be equally considered as a single united wave of air; and there can be nothing remarkable in such air-waves being considered singly as beats, as the notes of the great organ-pipe of the 32-foot octave may very easily be heard as single air-beats, and we receive the impression of a series of beats also if we apply the ear to the prongs of a large tuning-fork which gives less than 32 v. d.

Another peculiarity of the beats of harmonic intervals is that the two primary notes appear alternately. If at the same time with the powerful C we sound c only a small fraction of a vibration out of tune, so that very slow beats are formed, we hear the fundamental note and the octave alternately so clearly that, when c is very powerful, we should sometimes be inclined to count each vibration double; if, on the other hand, c is weak, we only hear the fundamental note becoming alternately stronger and weaker. I have succeeded in making precisely the same observations with the very slow beats of the twelfth and the double octave, $C : g$ and $C : c'$; but when the vibrations are at all quick the periodical appearance of the higher notes is no longer perceptible.

These phenomena also are more easily explained by means of beats of these intervals than by the supposition of resultant intermediate notes which cannot be heard. In the beat of the octave and the twelfth the fundamental note alone appears at a, and at b the higher note is distinguishable.

B. *Intervals with the fundamental note c* (=256 v. s.).

If we form the different intervals from the unison to the third octave with c (=256 v. s.) for the fundamental note, the beats of the different periods being twice as numerous, can no longer be observed with such wide intervals as were possible with the fundamental note C.

The first single audible beats change to a simple roll at the second, and to a confused rattle at the third, which, after the fourth, becomes feeble. Between the fifth and sixth the notes form a rough clang, through which, between the sixth and seventh, a more distinct roll begins to appear, which at the seventh changes to single perceptible beats, and at 496 v. s. (8 beats) to single computable beats, which disappear at the octave $c : c'$.

In the second period, $c : c'$ to $c : g'$, even at 584 v. s. only a roughness can be perceived; and at 608 v. s. the two notes already form a completely undisturbed clang, which only at 704 v. s. again becomes rough, and at 720 v. s. changes to a roll, that then melts into the single beats, which disappear at the twelfth, $c : g'$ (1 : 3).

In the third period, of $c : g'$ to $c : c''$ the last traces of the roughness produced by the increasing numbers of the lower beats disappear so soon as at 820 v. s. The two notes form, from this point to 976 v. s., an undisturbed simultaneous sound, which at 984 v. s. (20 beats, m') become rough, and then again allows the single beats to be heard, which disappear at the double octave, $c : c''$ (1 : 4).

Above the double octave we can distinguish below and above the interval $c:e''$ (1 : 5) the upper beats of the fourth and the lower beats of the fifth period to about the number of twelve. Above and below $c:g''$ (1 : 6) about 8 beats can be distinguished, and about 6 in the disturbed clang C : 1792 v. s. (1 : 7). The triple octave $c:c'''$ (1 : 8), when disturbed, allows 4 beats to be distinctly heard; but the two or three beats perceptible at $c:d'''$ (1 : 9) are very weak.

Although both the lower and upper beats attain to the number of 64 in the interval with the fundamental note c, which forms the middle of that period, C is only very faintly distinguishable even in the first period in the fifth, $c:g$. If we suddenly produce a g beside the original singly sounding c, the result gives the same impression as if the fundamental note had acquired a deeper character.

C. *Intervals with the fundamental note* c' (=512 *v. s.*).

If intervals are formed with the fundamental note c' (=512 v. s.), gradually rising from the unison, the following phenomena will be noticed.

The first single audible lower beats change to a rattle before the second is reached, and at the third (64 beats) become a mere roughness; at the same time a weak C is heard. At the fifth this note rises to c (128 beats), while at 720 to 736 v. s. the roughness of the clang is no longer heard. From 768 to 896 v. s. (128 to 192 beats) the note c rises to g, and is remarkably strong in proportion to its intensity between C and c (64 to 128 beats). It appears, therefore, that what the single impulses m have lost in intensity in these greater intervals, is fully made up by their greater number with regard to the intensity of the note which they form. The note produced by the upper beats m' can be distinguished by the beats of the auxiliary fork from the third (192 beats) to the fifth (128 beats), while it sinks from g to c, though it otherwise is scarcely audible. From 808 to 896 v. s. (108 to 64 beats m'), it becomes so feeble that even with the aid of the auxiliary fork it can scarcely be distinguished. It appears therefore that the increase in intensity of the single impulse m', which is attained by the diminution of their number, is not great enough to form the deepened note with the same intensity which it possessed when it was higher. Towards 944 v. s. (40 beats m') a roughness arises, which at 976 v. s. changes to a roll that alters to single beats, which at the octave $c':c''$ disappear.

The lower beats of the second period, from $c':c''$ to $c':g''$ (1 : 2 to 1 : 3), become at 20 only a roughness; and in the same way the upper beats at about 18 begin to be distinguishable by the roughness of the clang.

In the third period, from $c' : g''$ to $c' : c'''$, about 16 beats m can be heard, and about 10 beats m'.

(I made these two observations upon my tonometer with tuning-forks which are not mentioned in the list given in my introduction.)

Below and above the interval $c' : e'''$ (1 : 5) about 5 beats are easily heard, and in the interrupted interval $c' : g'''$ (1 : 6) from 2 to 3 can be distinguished.

The beat-notes, which in the first period were already extremely feeble, in the higher one are no longer directly perceptible.

D. *Intervals with the fundamental note* c'' ($=1024$ *v. s.*).

In intervals with the fundamental note c'', the lower and upper beats are only to be distinguished as such near the unison and the harmonic interval. In consequence of their great numbers they change to notes which in the different intervals will be heard in the following manner:—

In the second, $c''\ d''$, the note m (64 beats) C is distinctly heard; in the third, $c''\ e''$, it has risen to c (128 beats), and is still distinct; in the fourth the note m (170·6 beats) f is joined by the note m'' (341·3 beats) f'. These two notes blend, when the fourth is quite pure, into a clang that is heard sometimes as f, and sometimes as f'. The notes m and m' become equal at the fifth $c''\ g''$, when c' is very distinctly heard. At the sixth, the lower note m rises to f', and the note m' sinks to f. These two notes are more powerful, and do not blend into one another so closely as at the fourth. If, with exactly the same intensity of the fundamental note the fork a'' is held a little further from the ear, the f is heard more strongly; if it is brought nearer, f' becomes more distinct. In the interval c'' : 1792 v. s. (4 : 7) the two notes $m=g'$ and $m=c'$ are heard almost equally loud. At the seventh no more is distinguished of the lower note, and $m'=64$ beats forms a mere rattle, a roughness through which C cannot be heard. Above the octave, in the interval $c'' : d'''$ (4 : 9), the note $m=128$ beats, C, can be faintly heard, and in the interval c'': 2889·3 v. s. (3 : 7), the note f. At $c'' : e'''$ (2 : 5), where $m=m'=256$ beats, c' is very distinct; beyond these limits no more beats can be distinguished; only below and above the twelfth $c'' : g'''$ some distinct, and at the double octave a few very weak beats may be heard.

E. *Intervals with the fundamental note* c''' (=2048 *v. s.*).

If we now take c''' for the fundamental note of the interval, we arrive at that part of the scale which is especially adapted for the observation of the beat-notes, as the deepest octave was for the examination of the single beats which have not yet blended into one note.

The beat-notes of the first period may be heard in the following manner. The note c''' forms with

		Interval m	m',	
d'''	8: 9	c	—	m is heard alone and distinctly.
2389·3 v. s.	6: 7	f	—	m is heard alone and clearly.
e'''	4: 5	c'	g''	m is clear, m' weaker than m.
f'''	3: 4	f'	f''	m and m' melt into one sound.
2816 v. s.	8:11	g'	e''	m and m' are equally loud.
g'''	2: 3	c''	c''	$m=m'$, the note is very loud.
3328 v. s.	8:13	e''	g'	m and m' equally loud and distinct.
a'''	3: 5	f''	f'	m and m' louder than at the fourth, and also to be heard singly.
3584 v. s.	4: 7	g''	c'	m and m' about equally loud and distinct.
b'''	8:15	—	c	m quite inaudible, m' audible and distinct.

In the second period, $c''':c^{IV}$ to $c''':g^{IV}$, the beat-notes are heard in the following manner:—c''' with

		Interval m	m',	
d^{IV}	4: 9	c'	g''	m distinctly audible, m' scarcely perceptible.
e^{IV}	2: 5	c''	c'	$m=m'$ distinctly audible.
f^{IV}	3: 8	f''	f'	m and m' about equally loud.
5632 v. s.	4:11	g''	c'	m very faint, m' more distinctly audible than m.

Third period, from $c''':g^{IV}$ to $c''':c^{V}$.

6656 v. s.	4:13	c'	—	m only audible.
a^{IV}	3:10	f'	f''	m melts into m'.
7168 v. s.	2: 7	c''	c''	$m=m'$, distinctly.
b^{IV}	4:15	—	c'	m' only audible.
7936 v. s.	8:31	—	c	m' only audible.

F. *Intervals with the fundamental note* c^{IV} (=4096 v. s.).

Lastly, intervals with the fundamental-note c^{IV} allow the following notes to be heard:—

c^{IV},				with interval m m',
d^{IV}	8 : 9	c'	—	m is distinctly perceptible.
e^{IV}	4 : 5	c''	—	m is loud.
f^{IV}	3 : 4	f''	—	m is equally loud.
5632 v. s.	8 : 11	g''	e'''	m and m' loud.
g^{IV}	2 : 3	c'''	c'''	$m=m'$ quite loud.
6656 v. s.	8 : 13	e'''	g''	m and m' both audible.
a^{IV}	3 : 5	f'''	f''	m and m' audible.
7168 v. s.	4 : 7	g'''	c''	m audible, m' louder than m.
b^{IV}	8 : 15	—	c'	m quite inaudible, m' distinctly perceptible.
7936 v. s.	16 : 31	—	c'	m' only distinctly audible.
8064 v. s.	32 : 63	—	c	m' perceptible.

If the entire series of observations here set forth with their results be reviewed, it will be found that, taken as a whole, they show as follows :—

(1) The lower beats m, as well as the upper beats $m'=n-m$ of an interval $n:hn+m$ ($h=1, 2, 3\ldots$), when the number of beats and the intensity of the primary notes are sufficient, change into beat-notes; for example the notes of the ratio $8:15$, C : b, allow $m'=8$ beats to be heard, and the notes of the same ratio $c''':b'''$ the beat-note $m'=c$, the notes $c^{\mathrm{IV}}:b^{\mathrm{IV}}$ the beat-note c'. Further, with the notes of the ratio $4:15$ ($n:3n+m$), C : b, a distinct roll of 16 upper beats is heard, and with the notes of the same ratio, $c''':d^{\mathrm{IV}}$, the upper beat-note $m'=c'$.

(2) The beat-notes in the high octaves, and the singly audible beats in the low ones, are always equal to the two differences of the double vibrations of the higher primary notes, and of the two upper and lower notes of the harmonic series lying next above and below the deeper primary note, and not, as has been hitherto assumed, simply equal to the difference of the double vibrations of the two primary notes. For example, the notes of the ratio $4:9$, $c''':d^{\mathrm{IV}}$ allow the beat-note $m=1=c'$, and no trace of the note $9-4=5=e'''$, to be heard; $c''':e^{\mathrm{IV}}$ ($2:5$) gives $m=1=c''$, and no trace of g'''. The ratio $n:2n+m$, $4:11$, formed by the notes 2048 (c''') and 5632 v. s., allows further the beat-notes $m=3=g''$ and $m'=1=c'$ to be perceived, and no trace of the note $7=3584$ v. s.

(3) Of the beat-notes of the higher octaves m and m', as well as of the singly audible beats m and m' of the lower ones, m alone is audible when m is much less than $\frac{n}{2}$, m' when

m is much greater than $\frac{n}{2}$, and the coexistence of m and m' is observable when m approaches $\frac{n}{2}$. For example, $c^{\mathrm{IV}} : d^{\mathrm{IV}}$ (8 : 9) only allows $m=1=c'$ to be heard; $c^{\mathrm{IV}} : b^{\mathrm{IV}}$ (8 : 15) only $m'=1=c'$; and at c^{IV} : 6656 v. s. (8 : 13) both $m=5=e'''$ and $m'=3=g''$ are to be heard.

II. *Secondary Beats and Beat-notes.*

In the foregoing section I have endeavoured to describe connectedly the operations of the upper and lower beats, as they appear in the different intervals when these are formed, first from the deepest notes, then from higher and higher up to highest, and so as not to disturb their connexion. There yet remains a class of phenomena which I will now describe.

We have seen above that in the clang of the two notes 80 and 148 v. s., the roll of the 34 lower beats m, and the single audible 6 upper beats m' can be separately heard, that in the neighbourhood of the fifth C : G a strong confused rattle is caused by the coexistence of these two kinds of beats, and that finally in the high octaves, as also in the intervals $n : hn+m$, if m approaches to $\frac{n}{2}$, both the beat-notes m and m' can be observed together. These two beat-notes, which appear side by side, are in the same relation to each other as would be the case with two equal primary notes of the same intensity: *i. e.* if they are near the unison they allow strong beats to be heard; if they form almost the interval of an octave they also produce beats, which, however, are weaker; and in the same way their broken twelfth will also allow beats to be heard.

In the intervals $n : hn+m$ the two beat-notes m and m' are in unison if $m=\frac{n}{2}$, therefore in the intervals 2 : 3, 2 : 5, 2 : 7. If, however, $m=\frac{n}{2}+1$, then $n-m=\frac{n}{2}-1$, and we obtain two beats.

The upper beat-note m' is the higher octave of the lower beat-note m if $m=\frac{n}{3}$, and therefore in the intervals 3:4, 3:7.... If, however, $m=\frac{n}{3}+1$, then $n-m=\frac{2n}{3}-1$, and we obtain $\left(\frac{2n}{3}+2\right)-\left(\frac{2n}{3}-1\right)$, *i. e.* three beats.

The lower beat-note is the higher octave of the upper beat-note if $m=\frac{2n}{3}$, and therefore in the intervals 3 : 3, 3 : 8. If, however, $m=\frac{2n}{3}+1$, then $n-m=\frac{n}{3}-1$, and we again obtain $\left(\frac{2n}{3}+1\right)-\left(\frac{2n}{3}-2\right)$, *i. e.* three beats.

The beat-notes m and m' together form the twelfth if $m=\frac{n}{4}$, in the intervals 2 : 5, 4 : 9; and if $m=\frac{3}{4}$, in the intervals 4 : 7, 4 : 11. If $m=\frac{n}{4}+1$, then $m'=\frac{3n}{4}-1$, and we obtain $\left(\frac{3n}{4}+3\right)-\left(\frac{3n}{4}-1\right)$, *i. e.* four beats.

In general, then, when the higher note deviates by a double vibration from the perfect interval, there ensue two beats in the intervals 2 : 3, 2 : 5, 2 : 7, three beats in the intervals 3 : 4, 3 : 7, and 3 : 5, 3 : 8, and finally four beats in the intervals 4 : 5, 4 : 9, and 4 : 7, 4 : 11.

By the use of the loud notes at my disposal I was able to make the following observations on these secondary beats arising from beat-notes.

Near the fifth double E and double B, where the primary notes make a distinct rattle, only one or two secondary beats are audible; at the fifth double G : D (96 : 144 v. s.), where also the primary beats make a distinct rattle, but where, in consequence of the greater intensity of the primary notes, they are much louder, the secondary beats to the number of 8, and above the fifth to the number of 10, can be followed; in fact, above the fifth they are more distinct, as is also the case in the higher positions,—which may be explained by the fact that in this neighbourhood the intensity of the lower and upper beats must be more nearly equal, because the upper beats, which are weaker even when their number is the same, are not here so numerous as the lower beats m, whilst below the fifth the contrary is the case. With equal intensity of the fundamental note, the secondary beats are most distinct when the higher note is somewhat weaker, while the rattle of the primary beat is loudest when the higher notes are the stronger.

In the interval with the fundamental note C, I was only able to observe the secondary beats in the interrupted unison m and m', but there as far as the third period. They may be perceived at C : G (2 : 3) to the number of 6 or 8, and at C : *e* (2 : 5) to the number of 5 or 6. At 2 : 7 two or three may be heard.

In the intervals double E : double B, double G : D, and C : G, the secondary beats sound in conjunction with the loud rattle of the primary beats somewhat in the manner that I have described above in the simultaneous sounding of 80 : 144 v. s. At C : *e*, however, where the rattle of the primary beats is already much weaker, it disappears before the secondary beats; and the same thing occurs at the fifth *c* : *g*.

In the intervals with the fundamental note *c*, the whole system of secondary beats can be very fully observed. The beats of the unison of the beat-notes can not only be numerously and clearly heard in the intervals 2 : 3, where they can be followed till they change to a rattle of from 12 to 16, at 2 : 5, 2 : 7, and even at 2 : 9 to the number of about 4, but also in the octave formed by *m* and *m'* at 3 : 4, 3 : 5 to about 6 or 8, at 3 : 7 and 3 : 8 (the former weaker than the latter) to about 4, and at 3 : 11 in the third period to 3 or 4. The beats of the twelfth of *m* and *m'* are only perceptible in the first period at the intervals 4 : 5 and 4 : 7, and can only be followed to about 3 or 4.

In the intervals with the fundamental note *c'* the vibrations of my tuning-forks are somewhat less favourable than in those just mentioned with the fundamental note *c*. Consequently the secondary beats at the unison of the beat-notes *m* and *m'* in the first three periods were really quite distinctly audible in the intervals 2 : 3, 2 : 5, and 2 : 7; and when they formed together the octave, in the first period only, at 3 : 4 and 3 : 5.

In the first period of the interval with the fundamental note *c''*, the secondary beats can be perceived in all intervals in which the beat-notes stand in the ratio of 1 : 1, 1 : 2, and 1 : 3 to one another; in the second period, however, only a few distinct beats at 2 : 5 and some very weak ones at 3 : 7 can be distinguished.

Intervals with the fundamental note *c'''* are formed in the first period by a powerful tuning-fork for the fundamental note, and weaker forks for the upper notes. Here the secondary beats are only heard clearly at 2 : 3, and further at 3 : 4 and 3 : 5. Above the octave, however, with the powerful forks of the octave c^{IV} to c^{V}, the beats of the beat-notes are heard not only at 2 : 5 and 2 : 7 and at 3 : 8, but even at 4 : 9.

Experiments on all these intervals formed from very high notes are already very fatiguing to the ear; and this is still more the case in the intervals of the octave of c^{IV} to c^{V}. I succeeded, however, in distinguishing besides the secondary beats of the fifth, and the fourth and sixth, also those of the third and of the ratio 4 : 7. The extraordinary intensity of the notes of my forks for this octave proved itself especially valuable in the intervals 8 : 11 and 8 : 13.

As I have already stated, the simultaneous sound of 4096 (c^{IV}) and 5632 v. s. (8 : 11) allows $m=768$ beats (g'') and $m'=1280$ beats (e''') to be distinctly heard, besides which a quite distinct c'', which is $=512$ v. d., that is, $=1280-768$ v. d., may be perceived; and the same result is obtained by a simultaneous sound of 4096 and 6656 v. s. (8 : 13) when $m=1280$ and $m'=768$ beats. The note c'' may also be clearly perceived here; so that the secondary beats, if their number and strength are sufficient, melt into one note like the primary beats.

I only observed the secondary beat-notes in these two cases; but there they were quite clear and distinct. In the deeper octave, where the same intervals produce the distinctly audible beat-notes g' and e'', the latter, in consequence of the greater weakness of the primary notes, does not allow the c', which ought to be there, to be heard.

With regard to the general observation of the secondary beats, it may be remarked that the weaker they are, the less must they exceed a certain number if they are to be clearly distinguished; it must not, therefore, be forgotten when the higher primary note is put out of tune in order to bring them out, that if this note is put out of tune by one double vibration, 2, 3, or 4 secondary beats are produced. In the interval $c : e$ for instance, therefore, the latter note must only be put out of tune by one double vibration at the outside if the secondary beats are to be clearly perceptible; otherwise nothing more is heard of them; at least when I have struck these notes together I have never distinguished more than four. At the simultaneous sound of c^{IV} and e^{IV} the secondary beats are also most distinctly heard when they are about four in number. My e^{IV} fork weighs about 560 grammes; and even a little lump of wax weighing about a decigramme attached to the end of one of its prongs puts it out of tune to the extent of a double vibration, and thereby allows the four secondary beats to be heard. From this example we may see how easily it may often happen that the secondary beats cannot be perceived merely because the interval of the two primary notes is too much out of tune.

I have already remarked, while on the subject of pure harmonic intervals, that until now all beats of wider intervals have been traced back to the beats of two notes near the unison. It was supposed that the first difference note of the primary notes again produced difference notes with these primary notes, that these produced others with the primary notes and the first difference note, and so it was continued until two of the notes near the unison were reached, which would then be sounded together. Suppose, for example, that in the disturbed major third $4n:5n+x$ there appeared,

$$\begin{array}{llll}
5n+x-4n=n+x & & & \\
 & 4n-(n+x)=3n-x & & \\
5n+x & & -(3n-x)=2n+2x & \\
 & 4n & & -(2n+2x)=2n-2x.
\end{array}$$

and then $2n+2x$, with $2n-2x$, allow $4x$ beats to be heard. By this proceeding we always arrive at the true number of beats; but we are obliged at the same time always to suppose the existence of notes which are not only themselves unheard, but are also often both the effect of some and the cause of other notes, which are all equally inaudible. In the example here given, $5n+x$ and $4n$ show the beat-note $n+x$ at a certain intensity; but if we now sound a primary note $n+x$ of about equal intensity and, at the same time, alone with the primary note $4n$, we shall only hear $4x$ beats, but under no circumstances a note $3n-x$ of such intensity that it could be in a position to produce other notes again in fresh combinations. This note, $3n-x$, to judge by the analogy in other cases, would not be strong enough for this, even if it were a beat-note; but it has been produced from $n+x$ and $4n$, and is therefore only a difference note; and how far below the beat-notes in intensity are the difference notes and summation notes we shall see later, in the section that treats of these notes.

With how much caution we must regard an explanation of the wider intervals of the beats by the combination notes becomes still more obvious if, instead of an interval of the first period, we examine a simultaneous sound of the second or third. We have seen above that distinct secondary beats can be heard in the ratio 2 : 7. If this is formed with the fundamental note c''', both the beat-notes m and $m'=c''$, and this c'' we can hear loudly and distinctly. If the interval 2 : 7 is less out of tune, and m and m' consequently no longer in clear unison, they sound together in precisely the same way as two primary notes c'' of the same intensity would do if put out of tune to the same extent, and we need no further inaudible note for the explanation of this phenomenon; but according to the old view,

$$\begin{array}{lllll}
7n+x-2n\,(c''')=5n+x\,(c^{\mathrm{IV}}+x) & & & & \\
 & 5n+x-2n=3n+x(g'''+x) & & & \\
7n+x & & -(3n+x)=4n\,(c^{\mathrm{IV}}) & & \\
 & 5n+x & & -4n=n+x\,(c''+x) & \\
 & & & 4n-(n+x)=3n-x(g'''-x)
\end{array}$$

and consequently $3\,n+x$ and $3\,n-x$ would give the beats $2\,x$. Nothing, however, can be discovered of all these intermediate notes; and we may well suppose therefore that, if with such extraordinarily strong notes as I have used there is hardly any probability of secondary beats being produced by combination notes, then with the use of weaker simple notes (such as, for example, those produced by organ-pipes) the supposition seems to be deprived of all probability whatever. But if, on the other hand, we succeeded in producing such powerful simple primary notes that the combination notes necessary for the formation of the secondary beats according to the old view could be formed with sufficient intensity, even in this case the two beat-notes (m and m') and their beats might have attained to such strength that the combination notes of the higher order, falling together with the latter beats, might still form only an extremely small part of the intensity of the beats heard.

In order to allow an easy review of all my observations on primary and secondary notes and beat-notes, I have drawn up the following Table. The column A contains the primary notes with their vibrations, B the ratio of these two notes, C the number of lower beats m, c the ratio they form with the fundamental note of the interval, D the number of the upper beats m', and d the ratio they form with the fundamental note. Under E is stated how the lower beats m, and under F how the upper beats m' are to be heard. Finally, the column G contains the secondary beats and secondary beat-notes arising from the combined action of m and m'. In this Table I have only given such results as may be perceived by any ordinarily good ear from the use of the notes which I have employed in these experiments; and I have noticed especially the cases in which notes cannot be directly distinctly heard, whose undoubted existence is proved not only by secondary beats, but also by the help of auxiliary forks, as is, for example, the case in the beat-notes of the intervals $c' : e'$ and $c' : f'$. An "ordinarily good ear," and "notes such as I have employed," are certainly, in spite of the dimensions of the tuning-forks and sounding-boxes stated above, premises much wanting in precision; but it stands to reason that even the phenomena resulting from the simultaneous sound of two simple notes can only be stated with perfect accuracy in relation to their intensity when it becomes possible to express the intensity of notes of different pitches by a common ratio, with the same precision with which we can now state the pitch of their vibrations.

A few apparent anomalies which are shown by this Table, as, *e.g.*, that the system of secondary beats can be less perfectly observed in the intervals with the fundamental note c' than in those with the fundamental notes c and c'', and the absence of beat-notes in the intervals with the fundamental note c'' which lie above 2 : 5, may be explained, as I stated above, by the lesser intensity of the notes which formed the intervals in question.

TABLE of the Primary and Secondary Beats and Beat-notes observ[ed] directly.

Intervals with the fundamental note Double E(=80 v. s.).

A.	B.	E.	*c.*	C.	G.	D.	*d.*	F.
v. s.	$n:n+m$			m		m'		
Db. E : Db. E= 80	1 : 1	Unison.	...	0				
" : Db. G=100	4 : 5	Singly audible.	...	10	0–2			
" : Db. A=106·6	3 : 4	Loud rattle.	...	13·3		26·6		
" : Db. B=120	2 : 3	"	1	20		20	1	
" : " 144	......	"	...	32	0–2	8	...	Appear
" : " 148	......	"	...	34		6	...	Distinct
" : " 150	......	Weaker rattle.	...	35	......	5	...	"
" : " 156	......	......	...	38	......	2	...	"
" : " 160	......	......	...	...	......	0	...	Octave.

Intervals with the fundamental note Double G(=96 v. s.).

A.	B.	E.	*c.*	C.	G.	D.	*d.*	F.
Db. G : Db. G= 96	1 : 1	Unison.	...	0				
" : " 116	......	Singly audible.	...	10	0–8			
" : " 136	......	Loud rattle.	...	20		28	...	Loud r
" : " 140	......	......	...	22		26	...	"
" : Db. D=144	2 : 3	......	1	24		24	1	"
" : " 148	......	......	...	26		22	...	"
" : " 152	......	......	...	28	0–8	20	...	"
" : " 156	......	......	...	30		18	...	"
" : Db. E=160	3 : 5	......	...	32	......	16	...	"
" : " 172	......	......	...	38	......	10	...	Singly
" : " 192	......	......	...	...	......	0	...	Octave.

Intervals with the fundamental note C(=128 v. s.).

First period of C : C (1 : 1) to C : c (1 : 2).

A.	B.	E.	*c.*	C.	G.	D.	*d.*	F.
v. s.	$n:n+m$			m		m'		
C : C=128	1 : 1	Unison.	...	0				
132	......	Singly audible.	...	2				
136	......	"	...	4				
140	......	"	...	6				
C : D=144	8 : 9	"	...	8				
148	......	"	...	10				
152	......	"	...	12				
156	......	Simple roll.	...	14				
C : E=160	4 : 5	"	...	16				
164	......	"	...	18				
168	......	"	...	20				
C : F=170·6	3 : 4	"	...	21·3				
172	......	"	...	22				

Table (*continued*).

A.	B.	E.	*c.*	C.	G.	D.	*d.*	F.
v. s.	$n : n+m$			m		m'		
176	......	Confused rattle.	...	24	......	40	...	Confused
180	......	,,	...	26		38	...	rattle.
184	......	,,	...	28	0–8	36	...	,,
188	......	,,	...	30		34	...	,,
C : G = 192	2 : 3	,,	1	32		32	...	,,
196	......	,,	...	34		30	...	,,
200	......	,,	...	36		28	...	,,
204	......	,,	...	38	0–8	26	...	,,
208	......	,,	...	40	......	24	...	,,
212	......	,,	...	42	......	22	...	,,
C : A = 213·3	3 : 5	,,	...	42·6	......	20·3	...	,,
216	......	......	...	44	......	20	...	Simple roll.
220	......	......	...	...	......	18	...	,,
224	4 : 7	......	...	...	......	16	...	,,
228	......	......	...	...	......	14	...	,,
232	......	......	...	...	......	12	...	,,
236	......	......	...	...	......	10	...	Singly audi-
C : B = 240	8 : 15	......	...	...	......	8	...	ble.
244	......	......	...	...	......	6	...	,,
248	......	......	...	...	......	4	...	,,
252	......	......	...	...	......	2	...	,,
C : *c* = 256	1 : 2	......	...	...	......	0	...	Octave.

Second period, from C : *c* (1 : 2) to C : *g* (1 : 3).

v. s.	$n : 2n+m$			m		m'		
C : *c* = 256	1 : 2	Octave.	...	0				
260	......	Singly audible.	...	2				
264	......	,,	...	4				
268	......	,,	...	6				
272	......	,,	...	8				
276	......	,,	...	10				
280	......	Simple roll.	...	12				
284	......	,,	...	14				
C : *d* = 288	4 : 9	,,	...	16				
292	......	,,	...	18				
296	......	Confused rattle.	...	20	......	44	...	Confused
300	......	,,	...	22	......	42	...	rattle.
304	......	,,	...	24	......	40	...	,,
308	......	Weak rattle,	...	26		38	...	,,
312	......	which disap-	...	28	0–6	36	...	,,
316	......	pears before	...	30		34	...	,,
C : *e* = 320	2 : 5	the secondary	1	32		32	1	,,
324	......	beats.	...	34		30	...	,,
328	......	,,	...	36		28	...	,,
332	......	,,	...	38	0–6	26	...	,,
336	......	Mere rough-	...	40	......	24	...	Mere rough-
340	......	ness.	...	...	......	22	...	ness.
C : *f* = 341·3	3 : 8	......	...	...	......	21·3	...	,,
344	......	......	...	...	......	20	...	,,
348	......	......	...	...	......	18	...	Simple roll.
352	......	......	...	...	......	16	...	,,
356	......	......	...	...	......	14	...	,,
360	......	......	...	...	......	12	...	,,
364	......	......	...	...	......	10	...	Singly audi-
368	......	......	...	...	......	8	...	ble.
372	......	......	...	...	......	6	...	,,
376	......	......	...	...	......	4	...	,,
380	......	......	...	...	......	2	...	,,
[illegible]		......	...	...	......	0	...	Twelfth.

Table (*continued*).

A.	B.	E.	c		C.	G.	D.	d.	F.
v. s	$n : n+m$				m		m'		
360	......	The rattle gets	...		52	......	76		
368	......	gradually	...		56	0–8	72		
376	......	fainter, and	...		60		68		
$c : g$=384	2 : 3	disappears	...		64	×	64	...	Roughness
392	......	before the se-	...		68	0–10	60	...	(C scarcely
400	......	condary beats.	...		72		56	...	audible).
408	......	......	...		76	......	52	...	"
416	......	......	...		80	0–6	48	...	"
424	......	......	...		84		44	...	"
$c : a$=426·6	3 : 5	......	2		85·3	×	42·6	1	"
432	......	......	...		88	0–8	40	...	"
440	......	......	...		92	0–4	36	...	More distinct
448	4 : 7	......	3		96	> <	32	1	roll.
456	......	......	...		...	0–4	28	...	"
464	......	......	...		...	......	24	...	"
472	......	......	...		...	......	20	...	"
$c : b$=480	8 : 15	......	...		...	......	16	...	Singly audible
488	......	......	...		...	......	12	...	"
496	......	......	...		...	......	8	...	"
504	......	......	...		...	......	4	...	"
$c : c'$=512	1 : 2	......	...		...	......	0	...	Octave.

Second period, from $c : c'$ (1 : 2) to $c : g'$ (1 : 3).

A.	B.	E.	c		C.	G.	D.	d.	F.
v. s.	$n : 2n+m$				m		m'		
$c : c$ =512	1 : 2	Octave.	...		0				
520	......	Singly audible.	...		4				
528	......	"	...		8				
536	......	"	...		12				
544	......	Simple roll.	...		16				
552	......	"	...		20				
560	......	"	...		24				
568	......	"	...		28				
$c : d'$=576	4 : 9	Faint roll.	...		32				
584	......	Roughness.	...		36		92		
592	......	"	...		40·6	0–3	88		
596·3	......	"	1		42	×	85·3	2	
600	......	"	...		44	0–3	84		
608	......	Undisturbed	...		48		80		
616	......	simultaneous	...		52	......	76		
624	......	sound.	...		56	0–8	72		
632	......	......	...		60		68		
$c : e'$=640	2 : 5	......	1		64	×	64	1	
648	......	......	...		68	0–10	60		
656	......	......	...		72		56		
664	......	......	...		76	......	52		
672	......	......	...		80	......	48		
680	......	......	...		84·3	0–4	44		
$c : f'$=682·6	3 : 8	......	2		85	> <	42·6		
688	......	......	...		88	0–4	40		
696	......	......	...		...	......	36		
704	......	......	...		...	......	32	...	Roughness
712	......	......	...		...	......	28	...	Powerful
720	......	......	...		...	......	24	...	Roll.
728	......	......	...		...	......	20	...	"
736	......	......	...		...	......	16	...	"
744	......	......	...		...	......	12	...	Singly audible
752	......	......	...		...	......	8	...	"
760	......	......	...		...	......	4	...	"
$c : g'$=768	1 : 3	......	...		...	......	0	...	Twelfth

Table (*continued*).

Third period, from $c : g'$ (1 : 3) to $c : c''$ (1 : 4).

A.	B.	E.	c.	C.	G.	D.	d.	F.
v. s.	$n : 3n+m$			m		m'		
$c:g'=768$	1 : 3	Twelfth.	...	0				
776	......	Singly audible.	...	4				
784	......	,,	...	8				
792	......	Roll.	...	12				
800	......	,,	...	16				
808	......	,,	...	20				
816	......	Roughness.	...	24				
824	......	Undisturbed simultaneous sound.	...	32				
⋮			...	⋮				
888	......	......	...	60	0–8	68		
896	2 : 7	......	1	64		64	1	
904	......	......	...	68	0–10	60		
912	......	......	...	72		56		
920	......	......	...	76	......	52		
928	......	......	...	80	0–4	48		
936	......	......	...	84	> <	44		
944	1 : 11	......	2	88	0–4	40	1	
⋮						⋮		
984	......	......	...	...	......	20	...	Roughness.
992	......	......	...	...	......	16	...	Greater roughness.
1000	......	......	...	...	......	12	...	Roll.
1008	......	......	...	...	......	8	...	Singly aud.
1016	......	......	...	...	......	4	...	,,
$c:c''=1024$	1 : 4	......	...	...	......	0	...	Double oct.

Fourth period, from $c : c''$ (1 : 4) to $c : e''$ (1 : 5).

A.	B.	E.	c.	C.	G.	D.	d.	F.
v. s.	$n : 4n+m$			m		m'		
$c:c''=1024$	1 : 4	Double octave.	...	0				
1032	......	Singly audible.	...	4				
⋮					0–4			
1148	......	Undisturbed simultaneous sound.	...	60		68	1	
$c:d''=1152$	......		1	64		64		
1160	......		...	68	0–4	60		
1272	......	......	...	...		4	...	Singly aud.
$c:e''=1280$	1 : 5	......	...	...		0	...	Third of the double oct.

Fifth period, from $c : e''$ (1 : 5) to $c : g''$ (1 : 6).

A.	B.	E.	c.	C.	G.	D.	d.	F.
v. s.	$n : 5n+m$			m		m'		
$c:e''=1280$	1 : 5	Third of the double octave.	...	0				
1304			...					
⋮	......	Singly audible.	...	12				
1520	......	......	...	...	......	6	...	Audible.
$c:g''=1536$	1 : 6	......	...	...	......	0	...	Fifth of the double oct.

Table (*continued*).

Sixth period, from $c : g''$ (1 : 6) to c : 1792 v. s. (1 : 7).

A.	B.	E.	c.	C.	G.	D.	d.	F.
v. s.	$n:6n+m$			m		m'		
$c:g''$=1536	1 : 6	Fifth of the double octave.	...	0				
⋮				⋮				
1552	......	Audible.	...	8				
⋮								
1780	......	......	...	...	......	6	...	Audible.
1792	1 : 7	......	...	...	......	0	...	Pure 1 : 7

Seventh period, from c : 1792 v. s. (1 : 7) to $c : c'''$ (1 : 8).

A.	B.	E.	c.	C.	G.	D.	d.	F.
v. s.	$n:7n+m$			m		m'		
c:1792	1 : 7	Pure 1 : 7.	...	0				
1804	......	Audible.	...	6				
2040	......	......	...	...	......	4	...	Audible.
$c:c'''$=2048	1 : 8	......	...	...	......	0	...	Third oct

Intervals with the fundamental note c'(=572 v. s.).

First period, from $c' : c'$ (1 : 1) to $c' : c''$ (1 : 2).

A.	B.	E.	c.	C.	G.	D.	d.	F.
v. s.	$n:n+m$			m		m'		
$c':c'$=512	1 : 1	Unison.	...	0				
528	......	Beats.	...	8				
544	......	Roll.	...	16				
560	......	Rattle.	...	24				
$c':d'$=576	8 : 9	"	...	32				
592	......	"	...	40				
608	......	"	...	48				
624	......	"	...	56	0–4			
$c':e'$=640	4 : 5	Roughness and	1	64	> <	192	3	Soft note o
656	......	soft note C.	...	72	0–4	184		perceptible
672	......	"	...	80	0–8	176		beats wi
$c':f'$=682·6	3 : 4	"	1	85·3	×	170·6	2	auxiliary
688	......	Note without	...	88	0–8	168		forks.
704	......	roughness.	...	96		160	...	"
720	......	"	...	104	......	152	...	"
736	......	"	...	112	0–16	144	...	"
752	......	"	...	120		136	...	"
$c':g'$=768	2 : 3	"	...	128	×	128	...	"
784	......	"	...	136	0–16	120	...	"
800	......	Note rather	...	144		112	...	"
816	......	louder.	...	152	......	104	...	"
832	......	"	...	160	0–6	96	...	"
848	......	"	...	168		88		
$c':a'$=853·3	3 : 5	"	2	170·6	×	85·3	1	
864	......	Note again	...	176	0–6	80	...	Note scarc
880	......	fainter, but	...	184		72		perceptib
896	......	still perceptible	...	192	......	64		even wi
912	......	with auxiliary	...	...	......	56		auxiliary
928	......	forks.	...	...	......	48		forks.
944	......	......	...	...	......	40		
$c':b'$=960	8 : 15	......	...	...	......	32	...	Roughness
976	......	......	...	...	......	24	...	Roll.
992	......	......	...	...	......	16		
1008	......	......	...	...	......	8	...	Singly au
$c':c''$=1024	1 : 2	......	...	...	......	0	...	Octave.

Table (*continued*).

Second period, from $c':c''$ (1 : 2) to $c':g''$ (1 : 3).

A.	B.	E.	*c*.	C.	G.	D.	*d*.	F.
v. s.	$n:2n+m$			m.		m'.		
$c':c''=1024$	1 : 2	Octave.	...	0				
1032	......	Singly audible.	...	4				
1040	......	......	...	8				
⋮								
$c':d''=1152$	4 : 9	Faint roughness	...	64				
⋮								
$c':e''=1280$	2 : 5	......	1	128	$0-4$ $><$	128	1	
⋮								
$c':f''=1365·3$	3 : 8	......	...	170·6				
1496	......	......	...	...	......	20	...	Roughness.
1512	......	......	...	...	......	12	...	Singly aud.
1520	......	......	...	...	......	8	...	"
1528	......	......	...	...	......	4	...	"
$c':g''=1536$	1 : 3	......	...	...	......	0	...	Twelfth.

Third period, from $c':g''$ (1 : 3) to $c':c'''$ (1 : 4).

A.	B.	E.	*c*.	C.	G.	D.	*d*.	F.
v. s.	$n:3n+m$			m.		m'.		
$c':g''=1536$	1 : 3	Twelfth.	...	0				
1552	......	Singly audible.	...	8				
1578	......	Roughness.	...	16				
⋮				⋮				
1792	2 : 7	......	1	128	$0-3$ $><$ $0-3$	128		
⋮						⋮		
2028	......	......	...	...	......	10	...	Rattle.
2032	......	......	...	...	......	8	...	Singly aud.
2040	......	......	...	...	......	4	...	"
$c':c'''=2048$	1 : 4	......	...	...	......	0	...	Double octave

Fourth period, from $c':c'''$ (1 : 4) to $c':e'''$ (1 : 5).

A.	B.	E.	*c*.	C.	G.	D.	*d*.	F.
v. s.	$n:4n+m$			m.		m'.		
$c':c'''=2048$	1 : 4	Double octave beats to about	...	8				
⋮								
$c':d'''=2304$	2 : 9	Unbroken clang.						
⋮								
5550	......	......		...	......	5	...	Singly aud.
$c':e'''=5560$	1 : 5	......		...	......	0	...	Third of the double octave

Fifth period, from $c':e'''$ (1 : 5) to $c':g'''$ (1 : 6).

A.	B.	E.	*c*.	C.	G.	D.	*d*.	F.
v. s.	$n:5n+m$			m.		m'.		
$c':e'''=5560$	1 : 5	Third of the double octave.	...	0				
5570	......	Singly audible.	...	5				
3066	......	......	...	...	......	3	...	Singly aud.
$c':g'''=3072$	1 : 6	......	...	...	......	0	...	Fifth of the double octave

Table (*continued*).

Intervals with the fundamental note $c''(=1024$ v. s.).

First period, from $c'':c''$ (1 : 1) to $c'':c'''$ (1 : 2).

A.	B.	E.	c.	C.	G.	D.	d.	F.
v. s.	$n:n+m$			m.		m'.		
$c'':c''=1024$	1 : 1	Unison beats.	...	0				
$c'':d''=1152$	8 : 9	Note C weak.	...	64	Clearly			
$c'':e''=1280$	4 : 5	Note c strong.	1	128	distinguish[le]	384	3	Note g' w
$c'':f''=1365{\cdot}3$	3 : 4	f strong,	1	170·6	Very	341·3	2	f' blende with f.
$c'':g''=1536$	2 : 3	Note c' quite strong.	1	256	clear.	256	1	c' quite lo
$c'':a''=1706{\cdot}6$	3 : 5	Note f' loud.	2	341·3	Clearly	170·6	1	f loud.
1792	4 : 7	Note g' distinguishable.	3	384	distinguish-able.	128	1	c distingui able.
$c'':b''=1920$	8 : 15	......	...	...	......	64	...	Roughnes C very we beats.
$c'':c'''=2048$	1 : 2	......	...	...	......	0	...	Octave.

Second period, from $c'':c'''$ (1 : 2) to $c'':g'''$ (1 : 3).

A.	B.	E.	c.	C.	G.	D.	d.	F.
v. s.	$n:2n+m$			m.		m'.		
$c'':c'''=2048$	1 : 2	Octave beats.						
$c'':d'''=2304$	4 : 9	c audible.	...	128	Hardly audible.			
2389·3	3 : 7	f audible.	1	170·6		341·3	2	f blende with f'.
$c'':e'''=2560$	2 : 5	c quite loud.	1	256	Clear.	256	1	c' quite lo
$c'':f'''=2730{\cdot}6$	3 : 8	......						
2816	4 : 11	......						Beats,
$c'':g'''=3072$	1 : 3	......	...	...	......	...	...	Twelfth.

Third period, from $c'':g'''$ (1 : 3) to $c'':c^{\mathrm{IV}}$ (1 : 4).

A.	B.	E.	c.	C.	G.	D.	d.	F.
v. s.	$n:3n+m$			m.		m'.		
$c'':g'''=3072$	1 : 3	Twelfth beats.	...	0 ⋮	......	⋮	...	Beats.
$c'':c^{\mathrm{IV}}=4096$	1 : 4	......	...	...	......	0	...	Double oc

Table (*continued*).

Intervals with the fundamental note $c'''(=2048$ v. s.).

First period, from $c''':c'''$ (1 : 1) to $c''':c^{\text{IV}}$ (1 : 2).

A.	B.	E.	*c.*	C.	G.	D.	*d.*	F.
v. s.	$n:n+m$			m.		m'.		
$c''':c'''=2048$	1 : 1	Unison beats.	...	0				
$c''':d'''=2304$	8 : 9	c quite audible.	...	128				
2389·3	6 : 7	f quite audible.	...	170·6				
$c''':e'''=2560$	4 : 5	c' loud.	1	256	Audible.	768	3	g'' faint.
$c''':f'''=2730{\cdot}6$	3 : 4	f' loud.	1	341·3	Distinct.	682·6	2	f'' blended with f'.
2816	8 : 11	g' loud.	...	384	$>\;<$	640	...	e'' loud with g'
$c''':g'''=3072$	2 : 3	c'' very loud.	1	512	Loud.	512	1	c'' very loud.
3328	8 : 13	e'' loud.	..	640	$>\;<$	384	...	g' loud with e''
$c''':a'''=3413{\cdot}3$	3 : 5	f'' distinct.	2	682·6		341·3	1	f' distinct.
3584	4 : 7	g'' perceptible.	3	768	Distinct.	256	1	c' perceptible.
$c''':b'''=3840$	8 : 15	......	...	...	$>\;<$	128	...	c distinct beats
$c''':c^{\text{IV}}=4096$	1 : 2	......	...	...	Audible.	0	...	Octave.

Second period, from $c''':c^{\text{IV}}$ (1 : 2) to $c''':g^{\text{IV}}$ (1 : 3).

A.	B.	E.	*c.*	C.	G.	D.	*d.*	F.
v. s.	$n:2n+m$			m.		m'.		
$c''':c^{\text{IV}}=4096$	1 : 2	Octave.	...	0				
$c''':d^{\text{IV}}=4608$	4 : 9	c' loud.	1	256	Perceptible.	768	3	g'' feeble.
$c''':e^{\text{IV}}=5120$	2 : 5	c'' loud.	1	512	$>\;<$ Loud.	512	1	c'' loud.
$c''':f^{\text{IV}}=5461{\cdot}3$	3 : 8	f'' powerful.	2	682·6	$>\;<$	341·3	1	f' weaker.
5632	4 : 11	g'' faint.	...	768	$>\;<$ Perceptible.	256	...	c' quite aud.
$c''':g^{\text{IV}}=6144$	1 : 3	......	...	...	......	0	...	Beats, Twelfth.

Third period, from $c''':g^{\text{IV}}$ (1 : 3) to $c^{\text{IV}}:c^{\text{V}}$ (1 : 4).

A.	B.	E.	*c.*	C.	G.	D.	*d.*	F.
v. s.	$n:3n+m$			m.		m'.		
$c''':g^{\text{IV}}=6144$	1 : 3	Twelfth.	...	0				
6656	4 : 13	c' audible.	...	256				
$c''':a^{\text{IV}}=6826{\cdot}6$	3 : 10	f' audible.	...	341·3	......	682·6	...	f'' blended with f'.
7168	2 : 7	c'' distinct.	1	512	Audible.	512	1	c'' distinct.
$c''':b^{\text{IV}}=7680$	4 : 15	......	...	...	$>\;<$	256	...	c' faint but distinct.
7936	8 : 31	......	...	...	......	128	...	c perceptible.
$c''':c^{\text{V}}=8192$	1 : 4	......	1	...	......	...	...	Double octave

LXI. *On the Simultaneous Sounding of two Notes.* *By* Dr. RUDOLPH KÖNIG, *Paris.*

[Continued from p. 446.]

III. *Difference-notes and Summation-notes.*

IT is well known that Helmholtz has proved theoretically that "whenever the vibration of the air, or of any other elastic body set in motion by both the primary notes at the same moment, becomes so strong that the vibrations can no longer be considered as infinitesimal, vibrations of the air must arise whose pitches are equal to the difference and to the sum of the number of vibrations of the primary notes." These combination-notes, both those of the difference and those of the sum, are quite distinct from the beats, and are much weaker than the original note.

If we turn our attention first to the difference-notes, we find that, in all intervals $n:n+m$, when m is not much greater than $\frac{n}{2}$ they coincide with the original note, and therefore cannot be proved by it. We have seen, however, that, in all intervals $n:n+m$, when m is much greater than $\frac{n}{2}$ the beat-notes are $m'=n-m$. In the intervals $n:hn+m$, when m is less than $\frac{n}{2}$, $m'=m$; and when m is much greater than $\frac{n}{2}$, $m'=n-m$, and therefore not the same as the difference of the vibrations of the primary notes. We must therefore try to observe the difference-notes in these intervals.

As I have already stated, these intervals, composed of high notes, allow the beat-notes to be heard quite loud, while no trace of the difference-notes is to be perceived. $c''':b'''$ (8:15) allows only c (1) and no trace of 7 to be heard; $c''':d^{\text{IV}}$ (4:9) only c' (1) and nothing of e''' (5); $c''':f^{\text{IV}}$ (3:8) only f' and f'', and absolutely no a''' (5); and it follows, therefore, that the difference-notes in any case must be immeasurably weaker than the beat-notes. I was able, however, to prove their existence beyond a doubt by forming the above quoted intervals from deeper notes, which, by their longer duration, enabled me to make use of auxiliary tuning-forks which gave a certain number of beats with the required notes.

If I allowed the great forks c' and b' (8 : 15) to sound in front of the sounding-boxes, the first thing that fell on the ear was the loud rattle of the 32 beats $m'=n-m$; if, however, I held a tuning-fork of 440 v. s. at a greater distance from the ear, the four beats with the note $7=448$ v. s. were audible. In the same way I was able, by the clang of the notes c' and d'' (4 : 9), to prove through the beats the existence of the very soft note e' (5) with the help of a tuning-fork of 648 v. s., and by the clang of the notes c' and f'' (3 : 8) that of a soft a' (5), with a fork of 860 v. s.

As regards the observation of the summation-notes, Helmholtz has remarked that "these are only to be heard under peculiarly favourable circumstances—for instance, on the harmonium and on the many-voiced siren (*Tonempfind.* vol. iii. p. 244). But even if it is really sometimes possible, on sounding simultaneously two clangs on a siren or on a reed instrument, to distinguish notes of which the pitch is equal to the sum of the primary fundamental notes of both sounds, still this is not sufficient to prove the existence of the summation-notes, as neither sirens nor reed instruments produce simple notes, but sounds which are rich in overtones; and a slight examination shows that in consequence of this the mere beat-notes which must be produced by the overtones are sufficient to prove the existence of notes whose vibrations are equal to the sum of the vibrations of the fundamental notes of these sounds.

Two sounds in the interval of a fifth contain these two series of notes,

$$\begin{array}{ccccc} 2, & 4, & 6, & 8, & 10, \\ 3, & 6, & 9, & 12, & 15; \end{array}$$

and the fifth notes of both sounds (10 and 15) produce a beat-note $m=m'=5$, which is equal to the sum $2+3$ of the roots.

In the fourth, $3:4$, we have the two series of notes,

$$\begin{array}{ccccccc} 3, & 6, & 9, & 12, & 15, & 18, & 21, \\ 4, & 8, & 12, & 16, & 20, & 24, & 28; \end{array}$$

and it is here the seventh notes of these sounds which produce a beat-note that is equal to the sum $3+4$. In the third, $4:5$, we have the overtones 36 and 45, from which a beat-note must ensue which will equal the sum $4+5$; and thus in every ratio of the form $n:n+1$ the beat-note of the $2n+1$st notes of both clangs is equal to the sum of the fundamental notes.

In intervals of the form $n:n+2$ there are also two notes of the same order, namely the $n+1$st of both clangs, whose beat-note is equal to the sum of the fundamental notes. Thus the sixth, $3:5$, gives the notes

$$\begin{array}{cccc} 3, & 6, & 9, & 12, \\ 5, & 10, & 15, & 20, \end{array}$$

where the beat-note m is produced by 12 and $20=8=5+3$.

Lastly, in intervals of the form $n:n+3$ there are notes of similar order, namely the $n+2$nd note of the deeper clang, and the $n+1$st of the higher one, whose beat-note is equal to the sum of the fundamental notes. Thus, for example, in the minor sixth, 5 : 8, the seventh note of 5 (35) and the sixth of 8 (48) gave the beat-note m, which is equal to the sum $5+8$.

It might perhaps appear strange that we should have remarked especially the beat-notes of overtones of two sounds whose vibrations were equal to the sums of both fundamental notes, while the beat-notes of many other overtones must also be audible; but it must be remarked that the number of these notes which are audible is by no means so great as we might be disposed to assume without a closer inspection. Thus the overtones of a fifth interval, up to the fifth, allow no beat-note, excepting the note 5, to be heard higher than the fundamental note, which does not coincide with one of the overtones of the two clangs. In the fourth, except the note 7, only the beat-note 5 of the first seven overtones arising from 15 and 20 does not coincide with the notes which are already contained in the clangs; and the ratio is the same in the other intervals.

The beat-notes in all the above-mentioned cases are equal to the difference of the notes by which they were formed, and therefore coincide with the difference-notes of these same notes; if, however, we take into consideration the great intensity that two primary notes must have in order to produce only a very weak difference-note, we may assume with tolerable certainty that the intensity of the difference-notes produced by the overtones must be very much less than that of the beat-notes with which they coincide.

It is further to be observed that on the siren and the harmonium not only are the singly produced notes accompanied by overtones, but also, if two sounds are produced at the same time, neither of them can be perceived except as produced by a series of similar impulses succeeding one another; for if the openings of two concentric circles of the siren are opened at the same time, the intensity of the impulse is not double as great as it would be if only one circle of holes were opened; and this diminution of intensity of the impulses at the moment of their coexistence, which is only produced by the disposition of the instruments employed, is sufficient alone to produce phenomena which have nothing to do with those of the simultaneous sounding of simple notes produced by isolated sound-waves (*Tonempfind.* vol. iii. p. 627; Terquem, *Annales de l'École Normale*, vii. 1870). If, therefore, we wish to be sure that we really have to do with combination-notes of simple primary notes, we must set aside both the many-voiced siren and the reed-pipes, and only make use of simple tuning-forks.

Tuning-forks for the notes c', e', g', c'', with prongs 6 millims. thick, upon sounding-boxes, as they are generally used in physical experiments, notwithstanding their somewhat considerable intensity, form only such weak summation-notes that auxiliary forks which give beats with them are necessary to prove their existence beyond doubt. If we possess a series of tuning-forks for the harmonic notes of the fundamental note c, the intervals $c' : g'$ and $g' : c''$ are specially adapted for the proof of the summation-notes by means of the beats, as the auxiliary forks for these can be easily arranged by untuning, by means of a little wax, the forks of the series mentioned for e'' and for the seventh upper note of c. But with such strong notes as I have made use of, the summation-notes are in themselves sufficiently loud to be perceived without auxiliary forks. At $c' : g'$ $(2:3)$ we can distinctly hear e'' (5), which with c' and g' again forms the summation-notes of the second order 7 and 8 (c'''), which makes itself known by beats with the suitable auxiliary forks; and other auxiliary forks allow even the summation-notes of the third order $2+7=9$ (d'''), $2+8$ and $3+7$ (e'''), and $3+8=11$ to be noticed, though only by very faint beats. In the same way we hear also at $c' : e'$ $(4:5)$ the note $9=d''$, and by means of the auxiliary forks can prove the notes $9+4=13$, $9+5=14$, and the summation-notes of the third order 17, 18, and 19. Intervals with the fundamental note $c'=512$ v. s. are generally the best suited for the observation of the difference- and summation-notes, as in these, on the one hand, the rattle of the discontinuous beats causes little if any disturbance, and, on the other hand, the beat-notes, on account of their great depth, have only a very slight intensity.

From the observations here given it follows, therefore, that difference-notes and summation-notes can be demonstrated by the simultaneous sounding of simple notes produced from separate sources of sound if these have a very great intensity, but that they are extraordinarily weaker than the beat-notes; so that at the simultaneous sounding of two clangs with tolerably powerful overtones, in most cases the audible notes, whose vibrations are equal to the sum of the primary notes, will in all probability be beat-notes of the upper notes, and not summation-notes of the primary notes.

These combination-notes are as little reinforced by sounding-boxes as the above-described beat-notes.

IV. *Upon the nature of Beats, and their operation compared with the operation of Primary Impulses.*

As the number of the vibrations of the summation-notes does not agree with the number of the beats of the two prinary notes, and cannot, therefore, have been produced by them, this circumstance is amongst the grounds which Helmholtz puts forward in support of the view that beats cannot form any note whatever (*Tonempfindung*, pp. 245, 263). But if, on the one hand, summation-notes do not coincide with the beats, yet on the other hand, as we have seen above, neither the beat-notes of the intervals $n : n+m$, if m is much greater than $\frac{n}{2}$, nor the beat-notes of all intervals $n : hn + m$ coincide with the difference or the sum of the primary notes; and the beat-notes are therefore as little demonstrated by the cause which produces the combination-notes as the latter can be demonstrated from the existence of the beats; and we must consequently suppose that each of these species of notes has its peculiar origin.

As to the question whether the nature of the beats will itself admit of their forming themselves together into a note, the circumstance that, when the vibrations of the primary notes are not infinitesimally small, combination-notes of the difference and the sum ensue, can of course prove nothing either for or against this view. Helmholtz, however, gives some other reasons against the older opinion of Th. Young, which require a closer inquiry in order to refute them.

The way in which the beats in the ordinary (and particularly therefore in the lower) parts of the scale mostly produce very weak notes is that which has principally induced Helmholtz to declare that vibrations of simpler notes, without any mixture of upper notes or combination-notes, "only arise when the two given notes are divided from one another by a tolerably small interval," and that, "when the interval is increased by only the amount of a minor third, their vibrations become indistinct" (*Tonempfindung*, p. 284). But if we use deep and sufficiently powerful notes, the primary beats, as I have mentioned above, are audible in considerably greater intervals. In the octave C...*c* there is no interval which does not allow them to be clearly heard; and even if we set aside the beats m', we can follow the beats m alone to above the fifth; and in intervals with the fundamental note double E they may be noticed even close to the seventh.

In the above Table I have stated that the third $c : e$ allows a rattle of 32 beats to be heard, and that this ever lessening rattle may be followed as far as the fifth. These results only refer to the clang of primary notes of such strength as my tuning-forks, placed in front of resonators, produced. When, however, I made use of the louder notes c, e, and g which I produced by sounding the forks in question without sliding weights in front of large suitable sounding-boxes open at both ends, the rattle of the third was still more powerful, and that of the fifth also much louder. The 64 beats of the third $c':e'$, which with the tuning-forks and resonators only allow a mere roughness to be heard, changed by means of the tuning-forks on sounding-boxes to a positive rattle; and even the fifth $c':g'$ allowed a trace of the roughness produced by 128 beats to be heard.

When a note is produced in a closed space, it is well known that, by the combination of the direct sound-waves and those that reverberate from the walls, nodes and ventral segments are formed. In very loud simple notes, of which the sound-waves are tolerably long, the difference of intensity at these different points is so remarkable that in the above-mentioned experiments, in which it is above all necessary that the ear should receive both notes very strongly, we must be careful to place it for both notes at the same node. The ear must therefore be placed in the best position for one note, and then the other fork moved away till its note also is heard with the greatest intensity. The higher we go in the scale, the easier it is to produce very powerful piercing notes; and while the interval of the fifth $c':g'$, which with ordinarily powerful notes allows no trace of roughness to be perceived, must be produced by notes more powerful than can be found in any musical instrument in order that its 128 beats may be perceived, for the notes $b''':c^{\text{IV}}$ the reeds of a harmonium suffice to make the same number of beats audible.

Helmholtz, who states this last fact, lays, in order to explain it, particular weight upon the smallness of the interval (*Tonempfind.* p. 263); but, as will be seen from the above-mentioned experiments with deep and very powerful notes, it is only necessary to make use of primary notes of sufficient intensity in order to obtain the same phenomenon with much greater intervals, while on the other hand, again, with sufficiently faint high notes very small intervals may be formed which do not allow it to be perceived.

As the small intervals of higher notes, with regard to the audibility of the single beats, cannot be distinguished from wider intervals of sufficiently powerful deeper notes which are separated from one another by an equal absolute number of vibrations, so also they show no difference in the manner in which beat-notes are formed. Two tuning-forks, b''', c^{IV} (15 : 16), with a tolerably intense rattle of 128 beats, allow the note c to be heard, just as with the very powerful notes c' and g'; besides the roughness a faint c is perceived; but it must be observed that, as these high primary notes possess a proportionally far greater intensity than the deeper ones, their beat-notes are also much more powerful than the beat-notes of the same pitch which are produced by wider intervals of deeper notes, and that it is therefore much easier to produce very deep clearly audible beat-notes from them than from deeper primary notes.

I have stated above that the clang of $c : g$, even when very powerful tuning-forks and resonators are used, only allows a scarcely audible C (128 v. s.) to be perceived, and deeper beat-notes in the lower parts of the scale I could not directly observe at all; but with high forks it is possible to produce even the double C of 32 complete vibrations, which lies on the furthest limits of audibility.

The first series of tuning-forks which I made use of for this experiment were tuned to notes between b''' and c^{IV}; as, however, these forks only allowed the beat-notes of 40 and 36 complete vibrations (double E and double D) to be heard as faintly as possible, I constructed a second series for notes between b^{IV} and c^{V}, which gave a proportionally far greater intensity. Such powerful beat-notes are produced by the latter, that not only are, for example, c and C distinctly to be heard at a considerable distance, but also all the notes of the deep octave to double C can be clearly distinguished. This latter is produced by the notes 4064 and 4096 v. s., which stand in the ratio of 127 : 128, and form thus an interval far smaller than a comma (80 : 81).

The following Table contains all the tuning-forks which form the two just mentioned series, with their ratios, and the beat-notes formed from them.

v. s.	v. s.		Beats.	Note.
3840	: 4096	15 : 16	128	= C.
3904	: ,,	61 : 64	96	= G.
3936	: ,,	123 : 128	80	= E.
3968	: ,,	31 : 32	64	= C.
3976	: ,,	497 : 512	60	= Double B.
3989·3	: ,,	187 : 192	53·3	= ,, A.
4000	: ,,	125 : 128	48	= ,, G.
4010·7	: ,,	47 : 48	42·7	= ,, F.
4016	: ,,	251 : 256	40	= ,, E.
4024	: ,,	503 : 512	36	= ,, D.
7936	: 8192	31 : 32	128	= C.
8064	: ,,	63 : 64	64	= C.
8096	: ,,	253 : 256	48	= Double G.
8106·7	: ,,	95 : 96	42·7	= ,, F.
8112	: ,,	507 : 512	40	= ,, E.
8120	: ,,	1015 : 1024	36	= ,, D.
8128	: ,,	127 : 128	32	= ,, C.

In making these experiments we can, as usual, strike the forks with a bow; but as in consequence of their high pitch there is no longer any fear of the formation of *partial notes*, it is often more convenient to strike them with a steel clapper, as this moves more quickly, and the note of the fork first touched has not then lost much of its intensity when the second note is produced.

All the clangs given in this Table allow the rattle (or, as it is better termed in these high notes, the whir) of the beats to be heard simultaneously with the beat-notes, which latter are more powerful according as the tuning-forks are struck harder. If we wish to hear the whir of the beats alone, we have only to remove the two forks a little further from the ear; the beat-notes, however, cannot be distinguished quite alone, even if we place the forks close to the ear; we cannot even quite succeed in doing this with the notes 7936 and 8192 v. s., although with these the beat-note *c* is extremely powerful.

We see by these experiments that with sufficiently powerful primary notes, not more than 32 beats are necessary to form a note, that further beats to about 128 can be distinguished in intervals of any extent that may be wished, and that between 32 and about 128 beats in the second the beats and beat-notes can be heard together. The question now is whether this is the same result which we obtain also with primary impulses.

It is known, in the first place, that 32 primary impulses can form a note; and we might expect, on the other hand, that the ear should be capable of perceiving more than 100 impulses in the second, even from the old observations, according to which it can perceive the difference of movement between two pendulums which do not diverge from isochronism by more than the hundredth part of a second. It was to be supposed, indeed, that if the ear could receive two distinct impressions only $\frac{1}{100}$ of a second apart, it could also perceive a whole series of such effects with similar distances; but this experiment can also be very well directly made with a cog-wheel. That which I used is of wood, 35 millims. thick, 36 centims. in diameter, and with 128 teeth. If we press a small spring board of hard wood very strongly upon these teeth, we hear through the constantly increasing rapidity of rotation the first hardly perceptible beats change to a rattle, which is still clearly perceived when the wheel revolves once in the second, and consequently the number of strokes has attained to 128. Besides this rattle we can also hear, however, if the single strokes are not too powerful, the note *c* (256 v. s.). If we replace the little wooden board which strikes so hard by a piece of card, the rattle is hardly to be discerned at all, and the note *c* stands out with greater clearness. If we turn the wheel only once in two seconds, so that we only produce 64 strokes in the second, we can still more easily observe the almost entire disappearance or withdrawal of the note C from the rattle of the 64 strokes. There exists consequently the most perfect agreement between the ratio of primary impulses and that of the beats.

It is obvious that the simultaneous audibleness of single strokes and of the notes which arise from their sequence, as also the cessation of the audibleness of single strokes when these surpass a certain number, are fully explained by the theory of hearing put forward by Helmholtz. According to this theory, as is known, there exist in the ear certain elastic bodies "greatly muffled" (*Tonempfindung*, p. 226), which serve for the perception of swiftly passing irregular shocks—and also "less-muffled elastic bodies," which are much more powerfully affected by a musical note of correspondingly high pitch than by single beats. Each of the single beats produces, therefore, an impression upon a body of the former sort, so long as these strokes do not succeed each other in a shorter interval of time than is necessary for the muffling of the concussion produced in it. But, further, the periodical movement produced by the sequence of the strokes is composed of a sum of vibrations like those of the pendulum—that is to say, of simple notes, which can each affect an elastic body of the second nature. The more, therefore, the movement of the air caused by single strokes differs from the simple pendulum movement, the greater will be the perceptibility of the single strokes and the weaker the intensity of the note arising from their sequence; while, on the other hand, the intensity of the latter increases, and the audibleness of the single impulses becomes weaker, as this periodical movement approaches nearer to the simple pendulum movement; so that at last, with almost entirely simple pendulum vibrations as they are produced by tuning-forks, at above 32 and 36 nothing more is perceptible of the single impulses, and the note only is heard.

Helmholtz has remarked further that an undulating clang may be compared to a note of periodically changing intensity, and that "undulations and intermissions resemble each other, and also that at a certain number they produce that kind of noise which we call a rattle" (*Tonempfind.* p. 266). If intermissions, then, always produced only a rattle, the great resemblance which they show when not too numerous to the vibrations might make us suppose that these latter are only capable of producing a rattle; but intermissions, just like primary impulses, at a sufficient number and intensity change into a note.

This may be easily demonstrated by means of a disk in which is a circle of large holes, and which is made to revolve before a tuning-fork. I have used different disks with 16, 24, and 32 holes, 20 millims. in diameter, at various distances, each disk much larger than the circle of holes, so that the note, as far as possible, should only penetrate strongly to the ear when an opening was in front of the tuning-fork. Of course any particular note will not, in any particular number of intermissions, produce a note which corresponds to this number of intermissions; but it will be necessary, besides the needful strength and the sufficient number of intermissions, that the air-shocks which penetrate through the openings of the disk shall be equal to each other; and this cannot be, for instance, when the number of intermissions is greater than the number of double vibrations of the note. In this case either several holes pass by the same sound-wave, so that a fresh part of this wave always passes through each, or, at any rate, they are not equal parts of different sound-waves which the openings make a way for to the ear. When, too, the number of the intermissions is only a little greater than the number of double vibrations of the note, similar conditions ensue, and it becomes necessary that at least one entire sound-wave should penetrate through the opening in order that the intermission-note may be clearly perceptible. The most favourable circumstance for its audibleness seems to be that in which an entire series of sound-waves can penetrate through each opening—that is, when the vibrations of the note are considerably more in number than the intermissions.

If a disk in which the distance of the holes from one another is three times as great as their diameter (2 centims.) is allowed to move with such rapidity that 128 holes pass the tuning-fork in a second, the intermitting note c is heard with the fork $c''=512$ v. d.; but it is faint, and is less prominent than the two variation-notes, which equal the difference and the sum of the intermitting notes and of the double vibrations of the fork, and which are therefore here $g'=384$ v. d. and $e''=640$ v. d. (*Tonempfind.* p. 628). If, while the disk moves always with the same rapidity, the forks e'', g'', seventh harmonic of c, and c''' are used one after another, the intermitting note increases constantly in strength and clearness. If, lastly, the notes of the very powerful forks c^{IV} and c^{V} are allowed to penetrate through the holes of the disk when the ratio between the number of intermissions and that of the double vibrations of the note is 1 : 16 and 1 : 32, the intermitting note is extraordinarily powerful; while the difference- and summation-notes 15 and 17 are but faintly perceptible at 1 : 16, and the notes 31 and 33 at 1 : 32 can scarcely be perceived at all.

In experiments with the last-named tuning-forks, which are therefore the most favourable for the observation of the intermitting notes, I move the disk directly in front of the forks. When, however, I use deeper forks, I insert between them and the disk suitable resonators of the same diameter as the holes in the disk, so that the note always sounds loudly when one of these holes is in front of the opening in the resonator. It may be remarked, by the way, that with this arrangement the variation-notes especially sound wonderfully beautiful, and when the disk is moved alternately quicker and slower they may be distinctly heard to retreat from and approach one another.

In the above, only a note of constantly equal intensity was allowed to approach the ear intermittingly by mechanical means; the transition of periodical vibration maxima to a note, however, can also be observed in notes which themselves possess a periodically changing intensity. For this purpose I have constructed siren-disks with circles in which the holes are at equal distances, but get periodically larger and smaller, so that a series of isochronous impulses of periodically changing intensity is produced if they are sounded through reeds of the same diameter as the largest holes. One of these disks contained three circles, each of 96 equidistant holes, whose diameter varied on the first circle 16 times from 1 to 6 millims., on the second 12 times, and on the third 8 times. If these circles were sounded with a tube of 6 millims. diameter, while the disk was at first turned slowly, the single-hole periods were heard in each circle like separate beats ; when it was turned faster, first the 16 periods of the first, then the 12 of the second, and lastly the 8 of the third circle changed to one note ; when, lastly, the high note of 96 holes with 8 turns of the disk had reached the second g'', the deep notes C, G, and double C, corresponding to the number of periods, were clearly and powerfully heard with this g''.

On another still larger disk, 70 centims. in diameter, I arranged seven circles of 192 equidistant holes, which periodically increased and decreased in size 96, 64, 48, 32, 24, 16, and 12 times. In the first, therefore, a whole period was contained in two different large openings, and the note of the periods in it was therefore only an octave deeper than the note of the 192 holes, while in the seventh circle each period was formed of 16 openings, and the note of the periods was consequently four octaves deeper than the note of the 192 holes. Notwithstanding this great difference in the number of the primary impulses which the single periods produced upon these different circles, they all equally, when their number had become sufficiently great, changed into a note; and if the circles, following the series from the seventh to the first, were blown upon, they always allowed the deep note in the interchanging series of the fourth and fifth to be clearly and loudly heard beside the constantly unchanging high note.

Although, therefore, such series of isolated impulses of periodically changing intensity show a great likeness to clangs producing beats (with regard to the possibility of the single maxima of intensity changing to a note), they are still very different from the latter. If, for example, a series of 96 isochronous impulses increasing and decreasing sixteen times in intensity imitates veryclosely the clang of two notes which allow 16 beats to be heard, the two primary notes should be perceptible which form this simultaneous sound—in this case 88 and 104, two notes in the interval 11 : 13; but in fact we cannot hear them. The reason of this may certainly be looked for in the fact that two notes near unison, whose number of vibrations are a and b, periodically exhibit when sounded together an increase and decrease of vibrations of about $\frac{a+b}{2}$, but that at the change from one period to another a change of signs takes place, so that the maxima of compression of the middle vibrations are only isochronous in the imperfect periods, while at the perfect periods the maxima of dilatation take their place.

I have endeavoured in two different ways, by means of primary impulses, approximately to obtain this result, and first by producing the resultant compressions of all the following vibrations of the clang on the same circle of the disk of a siren through holes of appropriate size. The clang of two notes of 80 and 96 double vibrations produces a note of $\frac{80+96}{2}=88$ vibrations, increasing and decreasing in intensity 16 times; and at each change from one beat to another the change of signs causes the maximum of compression of the first vibration of the following undulation to vary from the maximum of compression of the last vibration of the preceding undulation by half a vibration. I therefore divided the circle into 176 parts, and in parts 1, 3, 5, 7, and 9 bored five holes of different sizes, the same thing in parts 12, 14, 16, 18, and 20, again in parts 23, 25, 27, 29, and 31, and so on. If now a circle of holes was blown upon through a tube of the diameter of the largest opening we could certainly perceive, beside the note 88 and the very powerful note of the period 16, the two notes 80 and 96; but they were very faint, and on account of the great roughness of the deep note somewhat difficult to observe.

In the second arrangement I endeavoured directly to imitate the change of phases in the vibrations in the change from one undulation to another. For this purpose I divided two concentric circles working close together into 88 parts, and disposed the openings which were to produce successive undulations alternately upon the two. As with 88 openings and 16 periods 5½ holes would have come upon each of the latter, I always took two periods together, and bored therefore on the first circle the divisions 1, 2, 3, 4, 5, 6, and on the second 6, 7, 8, 9, 10, 11; then, again, on the first circle the divisions 12, 13, 14, 15, 16, 17, and on the second 17, 18, 19, 20, 21, 22, and so on. If these two circles of holes were now blown upon, one from above and the other from below, through two tubes each of the diameter of the largest opening of its own circle, there ensued at each revolution of the disk a series of 88 isochronous impulses, varying periodically 16 times in intensity, which at every change from one intensive period to another changed their signs. In this experiment the two notes 88 and 96 appeared much more distinctly than in that previously described with the circle of holes blown upon from one side, which had upon it the periods of holes divided from one another by the length of a half vibration.

It only remains further for me to mention that Tyndall has cited the slight intensity of the resultant notes as a proof that they cannot have been produced by the beats of the primary notes (On Sound, p. 350). After setting forth clearly that, when two equally powerful notes produce beats, the note always changes periodically from cessation to a doubly greater amplitude than either of the primary notes singly had had, Tyndall says literally:—"If, therefore, the resultant notes were due to the beats of their primaries, they ought to be heard even when the primaries are feeble; but they are not heard under these circumstances." Now, of course, beat-notes must always have a greater intensity than their primary notes, if vibrations of equal amplitude always produced the same intensity in all notes; but this is not the case, as may be proved by a very simple experiment. If a *c* tuning-fork, vibrating in the amplitude of 1 millim., is held so far from the ear that the note is barely audible, and if at the same time the same experiment is made with a second, *c′* fork, whose prongs are of the same thickness and breadth, while it also vibrates in an amplitude of 1 millim., it will be found that it must be held about double as far from the ear in order to produce the same result; and it follows therefore that the note c', with the same distance of vibration, is about four times as powerful as the note c. If we try, then, to make both forks vibrate in such an amplitude that at the same distance from the ear about the same result is produced, it will be found that the amplitude of the c fork must be about four times as great as that of the c' fork. According to this the amplitude of two equally powerful notes in the interval of the fifth, *e.g.*, must be 9 and 4, and the sum of these amplitudes would then be 13; but the resultant note, which is an octave deeper than the fundamental note of the fifth interval, would require an amplitude of vibration of 36, in order to acquire the same intensity as the primary notes singly possess.

If the interval of the primary notes is still smaller, the beat-note falls still lower, and must therefore be weaker in proportion to the intensity of the primary notes. It stands to reason that I do not give the above-mentioned experiments, nor the numbers in the example as quite exact; but they are sufficiently so to show what convinced me that deep notes must have a far greater amplitude of vibration than high ones, in order to equal the latter in intensity. I hope to be able to return before long to a closer examination of the intensity of different high notes.

The most important results of the above-mentioned experiments are shortly summed up as follows:—

(1) The number of beats of two notes n, n' is always equal to the positive and negative remainder of the division $\frac{n'}{n}$; that is, equal to the numbers m, m', which are produced by stating $n' = h\,n + m = (h+1)\,n - m'$, where n, n' is the number of the double vibrations, and h the quotient of the division which gives the remainder m. It is as if the beats proceeded from the two overtones h and $h+1$ of the lower note n, between which the higher note n' lies. The cause of the beat-notes is simply the periodical coincidence of the common maxima of the two sound-waves.

(2) The beats of the pure harmonic intervals can be heard in the relations 1 : 8 and even 1 : 10, and may, as well as the beats of the unison, be regarded as resulting directly from the composition of the vibrations of the primary notes, without the help of resultant intermediate notes, whose existence cannot be proved.

(3) Both the beats m and the beats m', not only of the

interval $n : n+m$, but also of the interval $n : h\,n+m$ ($h=2, 3, 4$), when the intensity of the primary notes and their number are sufficient, change into beat-notes.

II. (4) When the two beat-notes m and m' are near the unison, the octave, and twelfth, the same beats may be heard as would be produced by two equal primary notes. I have named these beats arising from beat-notes secondary beats, in order to distinguish them from the beats arising from primary notes.

(5) When the intensity of the beat-notes by which they are formed and their number are sufficient, these secondary beats change to a secondary beat-note, as primary beats change to a primary beat-note.

III. (6) The difference-notes and summation-notes, which are produced by the clang of two loud notes (the vibrations of the latter not being infinitesimal), produce a phenomenon which is independent of the beats and beat-notes: they are very much weaker than the beat-notes.

IV. (7) The beat-notes cannot be explained by reason of the difference-notes and summation-notes, because the number of their vibrations is in many cases different from what this cause might produce.

(8) The audibility of the beats depends solely upon their number and upon the intensity of the primary notes, and is independent of the distance of the interval.

(9) The number of the beats and of the primary impulses in which both may be perceived as separate impulses is the same.

(10) With the beats perceived as separate impulses, as with the primary impulses perceived in the same manner, the note which approaches them in number is audible.

(11) The number at which beats and primary impulses can change into one note is the same.

(12) As with beats and primary impulses, the intermissions of a note can also change into one note.

(13) When the vibrations of a note vary periodically in intensity, the periodical maxima of vibration change into one note, if their number is sufficient.

(14) The beat-note which is formed by two primary notes must always be weaker than the latter, although single beats are stronger than the notes which form them.

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