XVIII. *On the Reflexion and Refraction of Sound.* *By* G. Green, Esq., *of Caius College, Cambridge.*

[Read *December* 11, 1837.]

The object of the communication which I have now the honour of laying before the Society, is to present, in as simple a form as possible, the laws of the reflexion and refraction of sound, and of similar phenomena which take place at the surface of separation of any two fluid media when a disturbance is propagated from one medium to the other. The subject has already been considered by Poisson, (*Mém. de l'Acad.*, &c. Tome X. p. 317, &c.) The method employed by this celebrated analyst is one that he has used on many occasions with great success, and which he has explained very fully in several of his works, and recently in a digression on the Integrals of Partial Differential Equations. (*Théorie de la Chaleur*, p. 129, &c.) In this way, the question is made to depend on sextuple definite integrals. Afterwards, by supposing the initial disturbance to be confined to a small sphere in one of the fluids, and to be everywhere the same at the same distance from its centre, the formulæ are made to depend on double definite integrals; from which are ultimately deduced the laws of the propagation of the motion at great distances from the centre of the sphere originally disturbed.

The chance of error in every very long analytical process, more particularly when it becomes necessary to use Definite Integrals affected with several signs of integration, induced me to think, that by employing a more simple method we should possibly be led to some useful result, which might easily be overlooked in a more complicated investigation. With this impression, I endeavoured to ascertain how a plane wave of infinite extent, accompanied by its reflected and refracted waves, would be propagated in any two indefinitely extended media of which the surface of separation in a state of equilibrium should also be in a plane of infinite extent.

The suppositions just made simplify the question extremely. They may also be considered as rigorously satisfied when light is reflected. In which case the unit of space properly belonging to the problem is a quantity of the same order as $\lambda = \frac{1}{50,000}$ inch, and the unit of time that which would be employed by light itself in passing over this small space. Very often too, when sound is reflected, these suppositions will lead to sensibly correct results. On this last account, the problem has here been considered generally for all fluids whether *elastic* or *non-elastic* in the usual acceptation of these terms; more especially, as thus its solution is not rendered more complicated. One result of our analysis is so simple that I may perhaps be allowed to mention it here. It is this: If A be the ratio of the density of the reflecting medium to the density of the other, and B the ratio of the cotangent of the angle of refraction to the cotangent of the angle of incidence. Then for all fluids

$$\frac{\text{the intensity of the reflected vibration}}{\text{the intensity of the incident vibration}} = \frac{A - B}{A + B}.$$

If now we apply this to the reflexion of sound at the surface of still water, we have $A > 800$, and the maximum value of $B < \frac{1}{4}$. Hence the intensity of the reflected wave will in every case be sensibly equal to that of the incident one. This is what we should naturally have anticipated. It is however noticed here because M. Poisson has inadvertently been led to a result entirely different.

When the velocity of transmission of a wave in the second medium, is greater than that in the first, we may, by sufficiently increasing the angle of incidence in the first medium, cause the refracted wave in the second to disappear. In this case the change in the intensity of the reflected wave is here shown to be such that, at the moment the refracted wave disappears, the intensity of the reflected becomes exactly equal to that of the incident one. If we moreover suppose the vibrations of the incident wave to follow a law similar to that of the cycloidal pendulum, as is usual in the Theory of Light, it is proved that on farther increasing the angle of incidence, the intensity of the reflected wave remains unaltered whilst the phase of the vibration gradually changes. The laws of the change of intensity, and of the subsequent alteration of phase, are here given for all media, *elastic* or *non-elastic*. When, however, both the media are *elastic*, it is remarkable that these laws are precisely the same as those for light polarized in a plane perpendicular to the plane of incidence. Moreover, the disturbance excited in the second medium, when, in the case of total reflexion, it ceases to transmit a wave in the regular way, is represented by a quantity of which one factor is a negative exponential. This factor, for light, decreases with very great rapidity, and thus the disturbance is not propagated to a sensible depth in the second medium.

Let the plane surface of separation of the two media be taken as that of (yz), and let the axis of z be parallel to the line of intersection of the plane *front* of the wave with (yz), the axis of x being supposed vertical for instance, and directed downwards; then, if Δ and Δ_1 are the densities of the two media under the constant pressure P and s, s_1 the condensations, we must have

$$\begin{cases} \Delta\,(1+s) = \text{density in the upper medium,} \\ \Delta_1(1+s_1) = \text{density in the lower medium.} \end{cases}$$

$$\begin{cases} P(1+As) = \text{pressure in the upper medium,} \\ P(1+A_1 s_1) = \text{pressure in the lower medium.} \end{cases}$$

Also, as usual, let ϕ be such a function of x, y, z, that the resolved parts of the velocity of any fluid particle parallel to the axes, may be represented by

$$\frac{d\phi}{dx},\quad \frac{d\phi}{dy},\quad \frac{d\phi}{dz}.$$

In the particular case, here considered, ϕ will be independent of z, and the general equations of motion in the upper fluid will be

$$0 = \frac{ds}{dt} + \frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2},$$

$$0 = \frac{d\phi}{dt} + \gamma^2 s;$$

where we have

$$\gamma^2 = \frac{PA}{\Delta},$$

or by eliminating s

$$(1) \qquad \frac{d^2\phi}{dt^2} = \gamma^2 \left(\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2}\right).$$

Similarly, in the lower medium

$$(2) \qquad \frac{d^2\phi_{,}}{dt^2} = \gamma_{,}^2 \left(\frac{d^2\phi_{,}}{dx^2} + \frac{d^2\phi_{,}}{dy^2}\right),$$

where

$$s_{,} = \frac{-d\phi_{,}}{\gamma_{,}^2 dt}, \quad \text{and} \quad \gamma_{,}^2 = \frac{PA_{,}}{\Delta_{,}}.$$

The above are the known general equations of fluid motion, which must be satisfied for all the internal points of both fluids; but at the surface of separation, the velocities of the particles perpendicular to this surface and the pressure there must be the same for both fluids. Hence we have the particular conditions

$$\left.\begin{aligned} \frac{d\phi}{dx} &= \frac{d\phi_{,}}{dx} \\ As &= A_{,}s_{,} \end{aligned}\right\} \text{(where } x = 0\text{)},$$

neglecting such quantities as are very small compared with those retained, or by eliminating s and $s_{,}$, we get

$$(A) \qquad \left.\begin{aligned} \frac{d\phi}{dx} &= \frac{d\phi_{,}}{dx} \\ \Delta \frac{d\phi}{dt} &= \Delta_{,} \frac{d\phi_{,}}{dt} \end{aligned}\right\} \text{(when } x = 0\text{)}.$$

The general equations (1) and (2), joined to the particular conditions (A) which belong to the surface of separation (yz), only, are sufficient for completely determining the motion of our two fluids, when the velocities and condensations are independent of the co-ordinate z, whatever the initial disturbance may be. We shall not here attempt to give their complete solution, which would be complicated, but merely consider the propagation of a plane wave of indefinite extent, which is accompanied by its reflected and refracted wave.

Since the disturbance of all the particles, in any *front* of the incident plane wave, is the same at the same instant, we shall have for the incident wave

$$\phi = f(ax + by + ct),$$

retaining b and c unaltered, we may give to the *fronts* of the reflected and refracted waves, any position by making for them

$$\phi = F(a'x + by + ct),$$

$$\phi_{,} = f_{,}(a_{,}x + by + ct).$$

Hence, we have in the upper medium,

$$(4) \qquad \phi = f(ax + by + ct) + F(a'x + by + ct),$$

and in the lower one

$$(5) \qquad \phi_{,} = f_{,}(a_{,}x + by + ct).$$

These, substituted in the general equations (1) and (2), give

$$(6) \qquad \begin{aligned} c^2 &= \gamma^2(a^2 + b^2), \\ c^2 &= \gamma^2(a'^2 + b^2), \\ c^2 &= \gamma_{,}^2(a_{,}^2 + b^2). \end{aligned}$$

Hence, $a' = \pm a$, where the lower signs must evidently be taken to represent the reflected wave. This value proves, that the angle of incidence is equal to that of reflexion. In like manner, the value of $a_{,}$, will give the known relation of sines for the incident and refracted wave, as will be seen afterwards.

Having satisfied the general equations (1) and (2), it only remains to satisfy the conditions (A), due to the surface of separation of the two media. But these by substitution give

$$af'(by + ct) - aF'(by + ct) = a_{,,}f_{,}'(by + ct),$$

$$\Delta\{f'(by + ct) + F'(by + ct)\} = \Delta_{,,}f_{,}'(by + ct),$$

because $a' = - a$, and $x = 0$.

Hence by writing, to abridge, the characteristics only of the functions

$$(7) \qquad \begin{aligned} f' &= \tfrac{1}{2}\left(\frac{\Delta_{,}}{\Delta} + \frac{a_{,}}{a}\right)f_{,}', \\ F' &= \tfrac{1}{2}\left(\frac{\Delta_{,}}{\Delta} - \frac{a_{,}}{a}\right)f_{,}', \end{aligned}$$

or if we introduce θ, $\theta_{,}$, the angles of incidence and refraction, since

$$\cot\theta = \frac{a}{b},$$

$$\cot\theta_{,} = \frac{a_{,}}{b},$$

$$f' = \tfrac{1}{2}\left(\frac{\Delta_{,}}{\Delta} + \frac{\cot\theta_{,}}{\cot\theta}\right)f_{,}',$$

$$F' = \tfrac{1}{2}\left(\frac{\Delta_{,}}{\Delta} - \frac{\cot\theta_{,}}{\cot\theta}\right)f_{,}',$$

$$\text{and therefore } \frac{F'}{f'} = \frac{\dfrac{\Delta_{,}}{\Delta} - \dfrac{\cot\theta_{,}}{\cot\theta}}{\dfrac{\Delta_{,}}{\Delta} + \dfrac{\cot\theta_{,}}{\cot\theta}},$$

which exhibits under a very simple form, the ratio between the intensities of the disturbances, in the incident and reflected wave.

But the equations (6) give

$$\gamma^2\left(\frac{a^2}{b^2} + 1\right) = \gamma_{,}^2\left(\frac{a_{,}^2}{b^2} + 1\right);$$

and hence

$$\frac{\gamma}{\sin\theta} = \frac{\gamma_{,}}{\sin\theta_{,}},$$

the ordinary law of sines.

The reflected wave will vanish when

$$0 = \frac{\Delta_{,}}{\Delta} - \frac{\cot\theta_{,}}{\cot\theta};$$

which with the above gives

$$\cot\theta = \Delta\sqrt{\frac{\gamma^2 - \gamma_{,}^2}{(\gamma_{,}\Delta_{,})^2 - (\Delta\gamma)^2}}.$$

Hence the reflected wave may be made to vanish if $\gamma^2 - \gamma_{,}^2$ and $(\gamma\Delta)^2 - (\gamma_{,}\Delta_{,})$ have different signs.

For the ordinary elastic fluids, at least if we neglect the change of temperature due to the condensation, A is independent of the nature of the gas, and therefore

$$A = A_{,} \text{ or } \gamma^2\Delta = \gamma_{,}^2\Delta_{,}.$$

Hence

$$\tan\theta = \frac{\gamma}{\gamma_{,}},$$

which is the precise angle at which light polarized perpendicular to the plane of reflexion is wholly transmitted.

But it is not only at this particular angle that the reflexion of sound agrees in intensity with light polarized perpendicular to the plane of reflexion. For the same holds true for every angle of incidence. In fact, since

$$\gamma^2\Delta = \gamma_{,}^2\Delta_{,}; \quad \therefore \frac{\Delta_{,}}{\Delta} = \frac{\gamma^2}{\gamma_{,}^2} = \frac{\sin^2\theta}{\sin^2\theta_{,}},$$

and the formulæ (7) give

$$\frac{F'}{f''} = \frac{\dfrac{\sin^2\theta}{\sin^2\theta_{,}} - \dfrac{\tan\theta}{\tan\theta_{,}}}{\dfrac{\sin^2\theta}{\sin^2\theta_{,}} + \dfrac{\tan\theta}{\tan\theta_{,}}} = \frac{\tan(\theta - \theta_{,})}{\tan(\theta + \theta_{,})};$$

which is the same ratio as that given for light polarized perpendicular to the plane of incidence. (Vide Airy's *Tracts*, p. 356.)

What precedes is applicable to all waves of which the *front* is plane. In what follows we shall consider more particularly the case in which the vibrations follow the law of the cycloidal pendulum, and therefore in the upper medium we shall have,

$$(8) \qquad \phi = \alpha \sin (a x + b y + c t) + \beta \sin (-a x + b y + c t).$$

Also, in the lower one,

$$\phi_{,} = \alpha_{,} \sin (a_{,} x + b y + c t):$$

and as this is only a particular case of the more general one, before considered, the equation (7) will give

$$\alpha = \tfrac{1}{2} \left(\frac{\Delta_{,}}{\Delta} + \frac{a_{,}}{a} \right) \alpha_{,},$$

$$\beta = \tfrac{1}{2} \left(\frac{\Delta_{,}}{\Delta} - \frac{a_{,}}{a} \right) \alpha_{,}.$$

If $\gamma_{,} > \gamma$, or the velocity of transmission of a wave, be greater in the lower than in the upper medium, we may by decreasing a render $a_{,}$ imaginary. This last result merely indicates that the form of our integral must be changed, and that as far as regards the co-ordinate x an exponential must take the place of the circular function. In fact the equation,

$$\frac{d^2 \phi_{,}}{d t^2} = \gamma_{,}^2 \left\{ \frac{d^2 \phi_{,}}{d x^2} + \frac{d^2 \phi_{,}}{d y^2} \right\};$$

may be satisfied by

$$\phi_{,} = \epsilon^{-a_{,}' x} . B \sin \psi,$$

(where, to abridge, ψ is put for $b y + c t$) provided

$$c^2 = \gamma_{,}^2 (- a_{,}'^2 + b^2);$$

when this is done it will not be possible to satisfy the conditions (A) due to the surface of separation, without adding constants to the quantities under the circular functions in ϕ. We must therefore take, instead of (8), the formula,

$$(9) \quad \phi = \alpha \sin (a x + b y + c t + e) + \beta \sin (- a x + b y + c t + e_{,}).$$

Hence, when $x = 0$, we get

$$\frac{d\phi}{dx} = a\,\alpha \cos(\psi + e) - a\,\beta \cos(\psi + e_{,}),$$

$$\frac{d\phi}{dt} = c\,\alpha \cos(\psi + e) + c\,\beta \cos(\psi + e_{,}),$$

$$\frac{d\phi_{,}}{dx} = - a_{,}' B \sin \psi,$$

$$\frac{d\phi}{dt} = c\,B \cos \psi\,;$$

these substituted in the conditions (A), give

$$\alpha \cos(\psi + e) - \beta \cos(\psi + e_{,}) = -\frac{a_{,}'}{a} B \sin \psi,$$

$$\alpha \cos(\psi + e) + \beta \cos(\psi + e_{,}) = \frac{\Delta_{,}}{\Delta} B \cos \psi\,;$$

these expanded, give

$$\alpha \cos e - \beta \cos e_{,} = 0,$$

$$-\alpha \sin e + \beta \sin e_{,} = -\frac{a_{,}'}{a} B,$$

$$\alpha \cos e + \beta \cos e_{,} = \frac{\Delta_{,}}{\Delta} B,$$

$$\alpha \sin e + \beta \sin e_{,} = 0.$$

Hence, we get,

$$(10) \qquad 2\,\alpha \sin e = \frac{a_{,}'}{a} B,$$

$$2\,\alpha \cos e = \frac{\Delta_{,}}{\Delta} B,$$

$$2\,\beta \sin e_{,} = -\frac{a_{,}'}{a} B,$$

$$2\,\beta \cos e_{,} = \frac{\Delta_{,}}{\Delta} B,$$

and, consequently,

$$e = -e_{,}, \qquad \beta = \alpha,$$

and

$$\tan e = +\frac{a_{,}'\Delta}{a\,\Delta_{,}}.$$

This result is general for all fluids, but if we would apply it to those only which are usually called *elastic*, we have, because in this case $\gamma^2\Delta = \gamma_{,}^2\Delta_{,}$,

$$\tan e = \frac{a_{,}'\Delta}{a\Delta_{,}} = \frac{a_{,}'\gamma_{,}^2}{a\gamma^2}.$$

But generally

$$(11) \qquad c^2 = \gamma_{,}^2(-a_{,}'^2 + b^2) = \gamma^2(a^2 + b^2);$$

and therefore, by substitution,

$$\tan e = \frac{a_{,}'\gamma_{,}^2}{a\gamma^2} = \frac{\gamma_{,}\sqrt{\gamma_{,}^2 b^2 - (a^2 + b^2)\gamma^2}}{a\gamma^2} = \mu\sqrt{\mu^2\tan^2\theta - \sec^2\theta},$$

because $\mu = \frac{\gamma_{,}}{\gamma}$, and $\frac{b}{a} = \tan\theta$.

As $e = -e_{,}$ we see from equation (9), that $2e$ is the change of phase which takes place in the reflected wave; and this is precisely the same value as that which belongs to light polarized perpendicularly to the plane of incidence; (Vide Airy's *Tracts*, p. 362.) We thus see, that not only the intensity of the reflected wave, but the change of phase also, when reflexion takes place at the surface of separation of two elastic media, is precisely the same as for light thus polarized.

As $\alpha = \beta$, we see that when there is no transmitted wave the intensity of the reflected wave is precisely equal to that of the incident one. This is what might be expected: it is, however, noticed here because a most illustrious analyst has obtained a different result. (Poisson, *Memoires de l'Academie des Sciences,* Tome X.) The result which this celebrated mathematician arrives at is, That at the moment the transmitted wave ceases to exist, the intensity of the reflected becomes precisely equal to that of the incident wave. On increasing the angle of incidence this intensity again diminishes, until it vanish at a certain angle. On still farther increasing this angle the intensity continues to increase, and again becomes equal to that of the incident wave, when the angle of incidence becomes a right angle.

It may not be altogether uninteresting to examine the nature of the disturbance excited in that medium which has ceased to transmit a wave in the regular way. For this purpose, we will resume the expression,

$$\phi_{,} = B\epsilon^{-a'_{,}x} \sin \psi = B\,\epsilon^{-a'_{,}x} \sin (b\,y + c\,t)\,;$$

or if we substitute for B, its value given by the last of the equations (10); and for $a'_{,}$, its value from (11); this expression, in the case of ordinary elastic fluids where $\gamma^2 \Delta = \gamma_{,}^2 \Delta_{,}$, will reduce to

$$\phi_{,} = 2\,a\,\mu^2 \cos e\,.\,\epsilon^{-\frac{2\pi x}{\lambda}\sqrt{\frac{\mu^2 \sin^2\theta - 1}{\mu}}} \sin (b\,y + c\,t),$$

λ being the length of the incident wave measured perpendicular to its own front, and θ the angle of incidence. We thus see with what rapidity in the case of light, the disturbance diminishes as the depth x below the surface of separation of the two media increases; and also that the rate of diminution becomes less as θ approaches the *critical* angle, and entirely ceases when θ is exactly equal to this angle, and the transmission of a wave in the ordinary way becomes possible.

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