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CONDUCTED BY

SIR ROBERT KANE, LL.D. F.R.S. M.R.I.A. F.C.S.

SIR WILLIAM THOMSON, KNT. LL.D. F.R.S. &c.

AND

WILLIAM FRANCIS, PH.D. F.L.S. F.R.A.S. F.C.S.

"Nec aranearum sane textus ideo melior quia ex se fila gignunt, nec noster vilior quia ex alienis libamus ut apes." JUST. LIPS. *Polit.* lib. i. cap. 1. Not.

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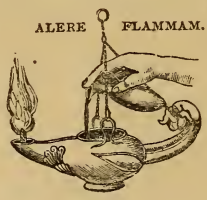
"Meditationis est perscrutari occulta; contemplationis est admirari perspicua . . . . . Admiratio generat quæstionem, quæstio investigationem, investigatio inventionem."—*Hugo de S. Victore.*

—"Cur spirent venti, cur terra dehiscat,
Cur mare turgescat, pelago cur tantus amaror,
Cur caput obscura Phœbus ferrugine condat,
Quid toties diros cogat flagrare cometas;
Quid pariat nubes, veniant cur fulmina cœlo,
Quo micet igne Iris, superos quis conciat orbes
Tam vario motu."

J. B. Pinelli ad Mazonium.



CONTENTS OF VOL. I.

(FIFTH SERIES).

NUMBER I.—JANUARY 1876.

Page

Dr. E. J. Mills on the First Principles of Chemistry 1
Mr. E. T. Newton on the Microscopic Characters of some Peculiar Forms of Coke. (Plate I.) 16
Mr. R. Mallet on "Mallet's Volcanic Theory tested" by the Rev. O. Fisher 19
M. C. Szily on the Second Proposition of the Mechanical Theory of Heat as deduced from the First 22
Mr. O. Heaviside on Duplex Telegraphy 32
Mr. J. W. L. Glaisher on the Representation of an Uneven Number as a Sum of four Squares, and as the Sum of a Square and two Triangular Numbers 44
Mr. F. Guthrie on Salt Solutions and Attached Water..... 49
Mr. S. H. Burbury on the Second Law of Thermodynamics in connexion with the Kinetic Theory of Gases 61
Mr. T. H. Marvin on the Production of Spectra by the Oxy-hydrogen Flame..... 67
Prof. R. Clausius on a new fundamental Law of Electrodynamics 69
Proceedings of the Royal Society:—
Mr. S. C. Tisley on a new Form of Dynamo-Magneto-Electric Machine 71
Mr. J. E. H. Gordon on the Determination of Verdet's Constant in Absolute Units 73
Prof. O. Reynolds on Rolling-Friction 75
Dr. Andrews on the Physical Properties of Matter in the Liquid and Gaseous States under varied conditions of Pressure and Temperature 78
On the Distribution of Magnetism in Circular or Elliptic Plates of Steel, by E. Duter 85
On the Temperature of the Upper Strata of the Atmosphere, by D. Mendeleef 86
On the Action of Flames in presence of Electrified Bodies, by M. Douliot..... 88

NUMBER II.—FEBRUARY.

Prof. E. Edlund's Experimental Proof that the Resistance to Galvanic Conduction is dependent on the Motion of the Conductor 89

Page

Prof. J. H. Cotterill on the Distribution of Energy in a mass of Liquid in a state of steady Motion 108
Mr. A. M. Mayer on the History of Young's Discovery of his Theory of Colours 111
Prof. H. How's Contributions to the Mineralogy of Nova Scotia. 128
Mr. O. Fisher on Mr. Mallet's Strictures on the Mathematical Test applied to his Theory of Volcanic Energy by Mr. O. Fisher 138
Mr. J. A. Fleming on the Polarization of Electrodes in Water free from Air 142
Notices respecting New Books :—
Mr. I. Todhunter's Elementary Treatise on Laplace's Functions, Lamé's Functions, and Bessel's Functions. 148
Mr. P. Frost's Solid Geometry 151
Proceedings of the Royal Society :—
Prof. W. G. Adams on the Change produced by Magnetization in the Electrical Resistance of Iron and Steel.—Preliminary Notice 153
Prof. W. G. Adams on the Action of Light on Selenium. 155
Proceedings of the Geological Society :—
Mr. T. Belt on the Drift of Devon and Cornwall 159
Mr. J. Gunn on the presence of the Forest-bed Series at Kessingland and Pakefield, in Suffolk 161
Mr. J. C. Ward on the Granitic, Granitoid, and associated Metamorphic Rocks of the Lake-district 161
Mr. W. H. Penning on the Physical Geology of East Anglia during the Glacial Period 164
Mr. A. Taylor on Denuding Agencies and Geological Deposition under the Flow of Ice and Water 166
Mr. R. A. Peacock on Historical and Personal Evidences of Subsidence beneath the Sea 168
Mr. H. Hicks on the Physical Conditions under which the Upper Silurian and succeeding Palæozoic Rocks were probably deposited over the Northern Hemisphere. 169
On the Recomposition of the Component Colours of White Light, by Professor J. Loudon 170
On the Transparence of Flames and of the Atmosphere, and on the Visibility of Scintillating Signal-lights, by E. Allard. 171
On some Properties of Gallium, by Lecoq de Boisbaudran .. 173
On the Spectrum of Gallium, by Lecoq de Boisbaudran 176

NUMBER III.—MARCH.

Mr. G. J. Stoney on Crookes's Radiometer 177
Mr. W. Ferrel on a controverted Point in Laplace's Theory of the Tides 182
Dr. C. Fromme on the Magnetism of Steel Bars 188

Page
Mr. W. Odling on the Formulation of the Paraffins and their Derivatives 205
Prof. R. Clausius on the Bearing of the Fundamental Law of Electrodynamics toward the Principle of the Conservation of Energy, and on a further Simplification of the former .. 218
Mr. H. M. Taylor on the Relative Values of the Pieces in Chess 221
Mr. C. J. Woodward on a new Form of Wave-apparatus suitable for the Lecture-room 229
Notices respecting New Books :—
The Chev. F. F. de Bruno's Théorie des Formes Binaires. 231
Proceedings of the Royal Society :—
Messrs. J. N. Lockyer and W. C. Roberts on the Absorption-Spectra of Metals volatilized by the Oxyhydrogen Flame 234
Messrs. W. De La Rue, H. W. Müller, and W. Spottiswoode on Stratification in Electrical Discharges *in vacuo* 239
Mr. W. N. Hartley on the Action of Heat on the Absorption-Spectra and Chemical Constitution of Saline Solutions 244
Mr. W. Crookes on Attraction and Repulsion resulting from Radiation 245
On the Attraction and Repulsion exerted by the Luminous and the Calorific Rays, by Dr. F. Neesen ; and on Crookes's Radiometer, by M. Poggendorff 250
On the Action of Heat in Magnetization, by L. Favé 253
On the Phenomena of Induction, by M. Mouton 255

NUMBER IV.—APRIL.

Lord Rayleigh on Waves 257
Profs. W. King and T. H. Rowney on the Serpentinite of the Lizard—its Original Rock-condition, Methylotic Phenomena, and Structural Simulations of Organisms. (Plate II.) 280
Dr. C. Fromme on the Magnetism of Steel Bars 293
Mr. G. J. Stoney on Crookes's Radiometer.—Part II. 305
Mr. W. Baily on a new Arrangement for the Micrometer of the Automatic Spectroscope 314
Prof. P. E. Chase on the Equilibrating Forces of the Solar System.—I. Mass and Position 315
Notices respecting New Books :—
Mr. W. D. Cooley's Physical Geography, or the Terraqueous Globe and its Phenomena 319
Proceedings of the Royal Society :—
Prof. W. G. Adams on the Action of Light on Tellurium and Selenium 322

Page

Prof. O. Reynolds on the Refraction of Sound by the Atmosphere 324
Proceedings of the Geological Society:—
Prof. A. C. Ramsay on the Island of Anglesey 326
Mr. J. A. Phillips on the Greenstones of Western Cornwall 327
The Rev. T. G. Bonney on Columnar, Fissile, and Spheroidal Structure 328
Mr. W. Ramsay on the influence of various substances in accelerating the precipitation of Clay suspended in water. 328
Mr. J. E. Marr on some Fossiliferous Cambrian Shales near Carnarvon 329
Mr. W. J. Harrison on the occurrence of the Rhætic Beds near Leicester 329
Mr. J. D. Kendall on Hæmatite in the Silurians 330
On the Spectra of Nitrogen and the Alkali-metals in Geissler tubes, by G. Salet 331
On certain remarkable points in Magnets, by R. Blondlot .. 333
On the Metallic Reductions produced in Capillary Spaces, by M. Becquerel 335
On the Chemical Actions produced by the Discharges from an Induction-apparatus, by M. Becquerel 336

NUMBER V.—MAY.

Mr. R. Sabine on a Method of Measuring very small Intervals of Time 337
Dr. L. Bleekrode on Electromachines with Disks of Ebonite. (Plate III.) 347
Mr. F. Guthrie on Salt Solutions with Attached Water 354
Mr. R. C. Nichols on the Proof of the Second Law of Thermodynamics 369
Mr. O. J. Lodge on some Problems connected with the Flow of Electricity in a Plane 373
Capt. C. O. Browne on the Determination of the Longitude of Cairo from Greenwich by the Exchange of Telegraph-Signals 390
Prof. Challis on a Theory of Mr. Crookes's Radiometer 395
Mr. R. Weber on a New Oxide of Sulphur and an analogous Selenium Substitution Product of the same 398
Notices respecting New Books:—
Prof. W. Dittmar's Manual of Qualitative Chemical Analysis 409
On the Influence of Temperature on Magnetization, by J.-M. Gaugain 411
Preliminary Note on Photographing the Least-Refracted Portion of the Solar Spectrum, by Capt. W. de W. Abney. 414
On the Occurrence of Phosphates in the Cambrian Rocks, by Mr. H. Hicks 415

NUMBER VI.—JUNE.

Page
Dr. R. König on the Simultaneous Sounding of two Notes .. 417
Mr. F. Guthrie on Salt Solutions and Attached Water..... 446
Don Enrique Serrano y Fatigati on Points in a Programme of Physics 455
Mr. B. O. Peirce, Jun. on the Induction Spark produced in breaking a Galvanic Circuit between the Poles of a Magnet 461
Mr. R. Colley on a Case of Work produced by the Electric Current 469
M. R. Pictet on the Application of the Mechanical Theory of Heat to the Study of Volatile Liquids: Simple Relations between the Latent Heats, Atomic Weights, and Tensions of Vapours. (Plate IV.) 477
Notices respecting New Books :—
Dr. R. S. Ball on the Theory of Screws: A Study in the Dynamics of a rigid body..... 489
On the Polar Auroras, by G. Planté 493
On Streams from Glaciers, by C. Knight 496

NUMBER VII.—SUPPLEMENT.

Prof. G. Tschermak on the Formation of Meteorites, and Volcanic Agency 497
Prof. P. E. Chase on the Nebular Hypothesis.—II. Interaction. 507
Dr. R. König on the Simultaneous Sounding of two Notes .. 511
Mr. L. Schwendler on the General Theory of Duplex Telegraphy 526
M. D. Mendelejeff on the Discovery of Gallium 542
Mr. W. H. Walenn on Unitation.—VI. Some of the Applications and Developments of the General Formula (continued). 546
Notices respecting New Books :—
Messrs. E. Lartet and H. Christy's 'Reliquiæ Aquitanicæ,' being contributions to the Archæology and Palæontology of Périgord, &c. Edited by Professor T. Rupert Jones, F.R.S., F.G.S. 550
Proceedings of the Geological Society :—
Dr. Dawson on the Phosphates of the Laurentian and Cambrian Rocks of Canada 558
Mr. H. B. Woodward on the Gravels, Sands, and other Superficial Deposits in the neighbourhood of Newton-Abbot..... 560
Mr. R. N. Worth on certain Alluvial Deposits associated with the Plymouth Limestone 560
Prof. A. C. Ramsay on the Physical History of the Dee, Wales 561
Mr. J. W. Judd on the Ancient Volcano of the District of Schemnitz, Hungary 562

Page
On the Effect of Thin Plates of Iron used as Armatures to Electromagnets, by J. Trowbridge 564
On "Feldspar" and "Feldstone" *versus* "Felspar" and "Felstone" 567
On the Displacement of the Lines in the Spectra of the Stars, produced by their Motion in Space, by Father Secchi 569
On a Simple and very Accurate Method of Tuning two Forks to Unison, by R. Spice 572

PLATES.

I. Illustrative of Mr. E. T. Newton's Paper on the Microscopic Characters of some peculiar forms of Coke.
II. Illustrative of Professors King and Rowney's Paper on the Serpentinite of the Lizard.
III. Illustrative of Dr. L. Bleekrode's Paper on Electromachines with Disks of Ebonite.
IV. Illustrative of M. R. Pictet's Paper on the Application of the Mechanical Theory of Heat to the Study of Volatile Liquids.

ERRATA.

Series 4, Volume 50.

Pge 405, line 27, *for* T = 34·8636 sec., *read* T = 34·836 sec. for the second experiment of Kohlrausch,

— 526, last line, *for* $\frac{1}{10n-}$ *read* $\frac{1}{10n-7}$

Series 5, Volume I.

Page 348, line 10 from bottom, *for* diameter *read* thickness.

— 349, line 9 from bottom, *for* active *read* inactive.

— 370, line 4 from bottom, *for* $\frac{dU}{dv}$ *read* $-\frac{dU}{dv}$

— — — 3 — *for* $\frac{dU'}{dv}$ *read* $-\frac{dU'}{dv}$

THE

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JOURNAL OF SCIENCE.

[FIFTH SERIES.]

APRIL 1876.

XXXII. *On Waves.* *By* Lord Rayleigh, *M.A., F.R.S.**

THE theory of waves in a uniform canal of rectangular section, in the case when the length of the wave is great in comparison with the depth of the canal, and when the maximum height of the wave is small in comparison with the same quantity, was given long ago by Lagrange, and is now well known. A wave of any form, subject to the above conditions, is propagated unchanged, and with the velocity which would be acquired by a heavy body in falling through half the depth of the canal. The velocity of propagation here referred to is of course relative to the undisturbed water. If we attribute to the water in the canal a velocity equal and opposite to that of the wave, the wave-form, having the same relative velocity as before, is now fixed in space, and the problem becomes one of *steady* motion. It is under this aspect that I propose at present to consider the question; and we will therefore suppose that water is flowing along a tube, whose section undergoes a temporary and gradual alteration in consequence of a change in the vertical dimension of the tube. The principal question will be how far the pressure at the upper surface can be made constant by a suitable adjustment of the velocity of flow to the force of gravity.

That the two causes which tend to produce variation of pressure at the upper surface act in opposition to each other is at once evident. If there were no gravity, the pressure would vary on account of the alteration in the velocity of the fluid. Since there must be the same total flow across all sections of

* Communicated by the Author.

the pipe, the fluid which approaches an enlargement must lose velocity, and the change of momentum involves an augmented pressure. On this account, therefore, there is an increased pressure at a place of enlargement and a diminished pressure at a contraction. On the other hand the effect of gravity is in the opposite direction, tending to produce a loss of pressure at the upper surface where that surface is high, and a gain of pressure where the surface is low. This effect of gravity is independent of the velocity; but the changes of pressure due to acceleration and retardation depend on the velocity of flow, and we can therefore readily understand that, with a certain definite velocity of flow, compensation may take place, at least approximately. When this happens, the condition of a free surface is satisfied, the constraint may be removed, and we are left with a stationary wave-form.

In the theory of long waves it is assumed that the length is so great in proportion to the depth of the water, that the velocity in a vertical direction can be neglected, and that the horizontal velocity is uniform across each section of the canal. This, it should be observed, is perfectly distinct from any supposition as to the height of the wave. If l be the undisturbed depth, and h the elevation of the water at any point of the wave, u_0, u the velocities corresponding to l, $l+h$ respectively, we have, by the condition of continuity,

$$u=\frac{lu_0}{l+h},$$

so that

$$u_0^2-u^2=u_0^2\,\frac{2lh+h^2}{(l+h)^2}.$$

By the principles of hydrodynamics, the increase of pressure due to retardation will be

$$\frac{\rho}{2}(u_0^2-u^2)=\frac{\rho u_0^2}{2}\,\frac{2lh+h^2}{(l+h)^2}.$$

On the other hand, the loss of pressure due to height will be $g\rho h$; and therefore the total gain of pressure over the undisturbed parts is

$$\left\{\frac{\rho u_0^2}{l}\,\frac{1+\dfrac{h}{2l}}{\left(1+\dfrac{h}{l}\right)^2}-g\rho\right\}h.$$

If now the ratio $h:l$ be very small, the coefficient of h becomes

$$\rho\left(\frac{u_0^2}{l}-g\right),$$

and we conclude that the condition of a free surface is satisfied provided $u_0^2 = gl$. This determines the rate of flow in order that a stationary wave may be possible, and gives of course at the same time the velocity of a wave in still water.

If we suppose the condition $u_0^2 = gl$ satisfied, the change of pressure is, to a second approximation,

$$\delta p = g\rho h \left\{ \frac{1 + \frac{h}{2l}}{\left(1 + \frac{h}{l}\right)^2} - 1 \right\} = -\frac{3}{2} \frac{g\rho h^2}{l},$$

which shows that the pressure is defective at all parts of the wave where h differs from zero. Unless, therefore, h^2 can be neglected, it is impossible to satisfy the condition of a free surface for a stationary long wave—which is the same as saying that it is impossible for a long wave of finite height to be propagated in still water without change of type. If, however, h be everywhere positive, a better result can be obtained with an increased value of u_0; and if h be everywhere negative, with a diminished value. We infer that a positive wave moves with a somewhat higher, and a negative wave with a somewhat lower velocity than that due to half the undisturbed depth.

Although a constant gravity is not adequate to compensate the changes of pressure due to acceleration and retardation in a long wave of finite height, it is evident that complete compensation is attainable if gravity be a function of height; and it is worth while to inquire what the law of force must be in order that long waves of unlimited height may travel with type unchanged. If f be the force at height h, the condition of constant pressure is

$$\frac{\rho}{2} u_0^2 \left\{ 1 - \frac{l^2}{(l+h)^2} \right\} = \rho \int_0^h f dh;$$

whence

$$f = -\frac{u_0^2}{2} \frac{d}{dh} \frac{l^2}{(l+h)^2} = u_0^2 \frac{l^2}{(l+h)^3},$$

which shows that the force must vary inversely as the cube of the distance from the bottom of the canal. Under this law the waves may be of any height, and they will be propagated unchanged with the velocity $\sqrt{fl}$, where f is the force at the *undisturbed level.*

The same line of thought may be applied to the case of a long wave in a canal whose section is uniform but otherwise arbitrary. Let A be the area of the section below the undis-

turbed level, b the breadth at that level. Then, as before,

$$(A+bh)u=Au_0$$

if h be small; whence

$$u_0^2-u^2=\frac{2bh}{A}u_0^2$$

Now by dynamics

$$u_0^2-u^2=2gh$$

if the upper surface be free; and thus

$$u_0^2=gA\div b,$$

which gives the velocity of propagation. In the case of a rectangular section we have the same result as before, since $A=bl$.

The energy of a long wave is half potential and half kinetic. If we suppose that initially the surface is displaced, but that the particles have no velocity, we shall evidently obtain (as in the case of sound) two equal waves travelling in opposite directions, whose total energies are equal, and together make up the potential energy of the original displacement. Now the elevation of the derived waves must be half of that of the original displacement, and accordingly the potential energies less in the ratio of 4 : 1. Since therefore the potential energy of each derived wave is one quarter, and the total energy one half of that of the original displacement, it follows that in the derived wave the potential and kinetic energies are equal.

We may now investigate the effect on a long wave of a gradual alteration in the breadth of the canal and the area of the section. The potential energy of the wave varies directly as the length, breadth, and square of the height; and, by what has been proved above, the same is true of the total energy. Now the length of the wave in various parts of the canal is obviously proportional to the velocity of propagation, viz. $\sqrt{\frac{A}{b}}$; and we may therefore write

$$E\propto\sqrt{\frac{A}{b}}\,.\,(\text{height})^2\,.\,b.$$

But when the alteration in the canal is very gradual, there is no sensible reflection and the energy of the wave continues constant; so that

$$\text{height}\propto A^{-\frac{1}{4}}b^{-\frac{1}{4}}.$$

In the case of a rectangular section,

$$\text{height}\propto l^{-\frac{1}{4}}b^{-\frac{1}{4}}.$$

These results are due to Green, Kelland, and Airy*. The same method may be even more easily applied to the sound-wave moving in a pipe of gradually varying section.

The theory of long waves may be applied in many cases to ascertain the effect on a stream of a contraction or enlargement of its channel. If the section of the channel up to the natural level of the stream be altered from A_0 to A, the equation of continuity gives

$$(A+bh)u=A_0u_0,$$

where b, the breadth at the surface of the water, is supposed not to vary with height. The condition of a free surface is

$$u_0^2-u^2=u_0^2\left\{1-\frac{A_0^2}{(A+bh)^2}\right\}=2gh,$$

or

$$\frac{A_0^2}{(A+bh)^2}=1-\frac{2gh}{u_0^2},$$

which shows that h can never exceed the height due to the velocity u_0, as is indeed otherwise obvious.

If the variations in A and b are small as well as gradual, and if we put $A=A_0+\delta A$, we find

$$h=\frac{\delta A}{b}\div\frac{gA_0}{bu_0^2}-1.$$

When the velocity u_0 is less than that of a free wave, $gA_0>bu_0^2$, and h has the same sign as δA; viz. a contraction of the channel produces a depression of the surface, and an enlargement an elevation. But if the velocity of the stream exceed that of a free wave, these effects are reversed, and an enlargement and contraction of the section entail respectively a depression and an elevation of the surface.

If the velocity of the stream is nearly the same as that of a free wave, a state of things is approached in which a wave can sustain itself in a stationary position without requiring a variation in the channel; and then the effects of such a variation are naturally much intensified.

We must not forget that these calculations proceed on the supposition that a steady motion is possible. It would appear that the motion thus obtained is unstable in the case where the velocity of the stream exceeds that of a free wave. If we suppose the upper surface to consist of a movable envelope, it would indeed be in equilibrium when disposed according to the law above investigated; but if a displacement be made and steady motion be conceived to be reestablished, the pressure of

* Stokes, Brit.-Assoc. Report on Hydrodynamics, 1846.

the fluid will be less than before if the displacement be downwards, but will be increased if the displacement be upwards; so that the forces brought to bear on the envelope are in both cases in the direction of displacement. The expression for the variation of pressure at the envelope is

$$\frac{\delta p}{\rho} = u_0^2 \frac{\delta A}{A_0} + \left(\frac{u_0^2 b}{A_0} - g\right) h.$$

The Solitary Wave.

This is the name given by Mr. Scott Russell to a peculiar wave described by him in the British-Association Report for 1844. Since its length is about six or eight times the depth of the canal, this wave is, to a rough approximation, included under the theory of long waves; but there are several circumstances observed by Mr. Russell which indicate that it has a character distinct from that of other long waves. Among these may be mentioned the very different behaviour of solitary waves according as they are positive or negative, viz. according as they consist of an elevation or a depression from the undisturbed level. In the former case the wave has a remarkable permanence, being propagated to great distances without much loss; but a negative wave is soon broken up and dissipated.

Airy, in his treatise on Tides and Waves, still probably the best authority on the subject, appears not to recognize any thing distinctive in the solitary wave. He says:—"We are not disposed to recognize this wave as deserving the epithets 'great' or 'primary,' . . . . and we conceive that ever since it was known that the theory of shallow waves of great length was contained in the equation $\frac{d^2X}{dt^2} = g\kappa \frac{d^2X}{dx^2}$, . . . the theory of the solitary wave has been perfectly well known." And again, "Some experiments were made by Mr. Russell on what he calls a *negative* wave—that is, a wave which is in reality a progressive hollow or depression. But (we know not why) he appears not to have been satisfied with these experiments, and has omitted them in his abstract. All the theories of our IVth Section, without exception, apply to these as well as to *positive* waves, the sign of the coefficient only being changed."

On the other hand, Professor Stokes says*:—"It is the opinion of Mr. Russell that the solitary wave is a phenomenon *sui generis*, in no wise deriving its character from the circumstances of the generation of the wave. His experiments seem to ren-

* British-Association Report.

der this conclusion probable. Should it be correct, the analytical character of the solitary wave remains to be discovered."

The theory of the solitary wave has been considered by Earnshaw (Camb. Trans. vol. viii.), who, distrusting what he calls analytical approximations, bases his calculation on a supposed result of experiment, namely that the horizontal velocity is uniform over each section. This, as we have seen, is the fundamental assumption in the theory of long waves; but when the length of the wave is moderate, such a state of things is impossible in a frictionless fluid which has been once at rest; for it involves molecular rotation. In fact if there be a velocity-potential, the horizontal velocity u satisfies Laplace's equation

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0,$$

and therefore cannot be a function of x without being also a function of y. The motion investigated by Earnshaw has therefore molecular rotation; and the rotation remains constant for each particle; otherwise the equations of fluid motion would not be satisfied. This is the explanation of the difficulty with which Earnshaw meets,—that while the necessary conditions are satisfied in the wave itself, there is discontinuity in passing from the wave to the undisturbed water. The discontinuity arises from the fact that, as there is no rotation outside the wave, it is necessary to suppose finite rotations imparted to the particles as the wave reaches and leaves them. It is evident that, except in the case of very long waves, u must be treated as a function of y as well as of x.

In considering the theory of long waves (reduced to rest by imparting an opposite motion to the water), we saw that it was impossible to satisfy the condition of a free surface if the height of the wave were finite. It occurred to me to inquire whether there might not be compensation in certain cases between the variation of pressure at the upper surface due to a finiteness of height, and the variation due to a departure from the law of uniform horizontal velocity proper to very long waves. It was conceivable that the surface-condition in the case of a wave of given finite height might be better satisfied by a moderate than by a very great wave-length. In this way I have obtained what seems to be a perfectly satisfactory approximate theory of the solitary wave.

If u and v be the horizontal and vertical velocities in a stream moving in two dimensions without molecular rotation, and ϕ, ψ the potential and stream functions, we have

$$\left.\begin{aligned} u &= \frac{d\phi}{dx} = \frac{d\psi}{dy}, \\ v &= \frac{d\phi}{dy} = -\frac{d\psi}{dx}. \end{aligned}\right\} \quad \ldots\ldots \quad (A)$$

Hence, if the bottom of the canal be taken for axis of x, we may take for u and v, since they satisfy Laplace's equation,

$$\left.\begin{aligned} u &= \cos\left(y\frac{d}{dx}\right)f(x) = f - \frac{y^2}{1.2}f'' + \frac{y^4}{1.2.3.4}f^{\mathrm{IV}} - \&c., \\ -v &= \sin\left(y\frac{d}{dx}\right)f(x) = yf' - \frac{y^3}{1.2.3}f''' + \ldots, \end{aligned}\right\} \quad (B)$$

where $f(x)$ is the slowly variable value of u at the bottom when $y=0$, and accents indicate differentiation with respect to x. The corresponding expression for ψ is

$$\psi = yf - \frac{y^3}{1.2.3}f'' + \frac{y^5}{1.2.3.4.5}f^{\mathrm{IV}} - . \quad . \quad (C)$$

This equation applies to the upper boundary, if we understand by ψ the there constant value of the stream-function, and gives us a relation between the ordinate of the boundary and the function f.

If p be the pressure at the upper surface, we have

$$-2\frac{p-\mathrm{C}}{\rho} = 2gy + u^2 + v^2,$$

where C is some constant. We will write for brevity,

$$u^2 + v^2 = \varpi - 2gy; \quad \ldots\ldots \quad (D)$$

and the object of the investigation is to examine how far it is possible to make ϖ constant by varying the form of y as a function of x. Since $u^2+v^2=(1+y'^2)u^2$, our equation becomes

$$yu = \sqrt{\frac{\varpi y^2 - 2gy^3}{1+y'^2}},$$

or, on substituting for u its value,

$$fy - \frac{y^3}{1.2}f'' + \frac{y^5}{1.2.3.4}f^{\mathrm{IV}} - \&c. = \sqrt{\frac{\varpi y^2 - 2gy^3}{1+y'^2}}.$$

Between this equation and (C), f may be eliminated by successive approximation; and we obtain as the relation between y and ϖ,

$$\psi\left\{1 - \frac{y^3}{3}\left(\frac{1}{y}\right)'' - \frac{y^5}{45}\left(\frac{1}{y}\right)^{\mathrm{IV}} - \ldots\right\} = \sqrt{\frac{y^2 - 2gy^3}{1+y'^2}}. \quad . \quad (E)$$

In this investigation y is regarded as a function of x, which

varies slowly, or (as we may put it) a function of ωx, where ω is a small quantity. If we agree to neglect the fourth power of ω, the third and following terms on the left-hand side of (E) may be omitted, and we obtain

$$\psi^2\left\{1+y'^2-\frac{2}{3}y^3\frac{d^2y^{-1}}{dx^2}\right\}=\varpi y^2-2gy^3,$$

or

$$\psi^2\left\{1-\frac{1}{3}y'^2+\frac{2}{3}yy''\right\}=\varpi y^2-2gy^3, \quad . \quad . \quad . \quad (F)$$

by which the value of ϖ is determined approximately in terms of the form of the upper surface. If we suppose ϖ constant and integrate (F) on that hypothesis, we shall obtain a form of upper surface for which the pressure varies very slightly, provided of course that the solution so obtained satisfies the suppositions on which the differential equation (F) is founded.

To integrate (F) we may write it in the form

$$\psi^2\left\{1+\frac{4}{3}y^{\frac{3}{2}}\frac{d^2y^{\frac{1}{2}}}{dx^2}\right\}=\varpi y^2-2gy^3,$$

or

$$\frac{d^2y^{\frac{1}{2}}}{dx^2}=\frac{3}{4\psi^2}(\varpi y^{\frac{1}{2}}-2gy^{\frac{3}{2}}-\psi^2y^{-\frac{3}{2}}),$$

which becomes a complete differential when multiplied by $2\frac{dy^{\frac{1}{2}}}{dx}dx$. Thus we find

$$\frac{1}{3}y'^2=\mathrm{C}y+\frac{\varpi y^2-gy^3}{\psi^2}+1,$$

C being the constant of integration. Suppose now that in the undisturbed parts of the canal the depth is l and the velocity u_0. Then

$$\varpi=u_0^2+2gl,$$

and

$$\psi=\int_0^l u_0dy=u_0l.$$

Substituting these, we get

$$\frac{y'^2}{3}=1+\mathrm{C}y+\frac{u_0^2+2gl}{u_0^2l^2}y^2-\frac{g}{u_0^2l^2}y^3. \quad . \quad . \quad (G)$$

In this equation g and l are given, while u_0 and C are at our disposal; and thus the cubic expression on the right may be made to vanish for $y=l$ and $y=l'$, where l' is the distance between the summit of the wave and the bottom of the canal. If

we substitute these values of y and eliminate C, we find

$$u_0^2 = gl' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (H)$$

as the relation between u_0 and l'. The constants C and u_0 being now determined so as to make y' vanish when $y=0$ and when $y=l'$, it will be found that the third root of the cubic is also l, so that our equation may be put into the form

$$y'^2 + \frac{3}{l^2 l'}(y-l)^2(y-l') = 0. \quad . \quad . \quad . \quad . \quad (I)$$

From this result it appears that there is only one maximum or minimum value of y (besides l); and since $y-l'$ is necessarily negative, it follows that the surface-condition cannot be satisfied to this order of approximation by a solitary wave of depression. Differentiating (I), we get

$$y'' = \frac{3(y-l)}{2l^2 l'}\{2l' + l - 3y\},$$

which shows that the points of zero curvature occur when $y=l$ and when $y = \frac{2l'+l}{3} = l + \frac{2}{3}(l'-l)$. Thus the curvature changes sign at two thirds of the height of the wave above the undisturbed level, and at these points only. The nature of the wave is sufficiently defined by (I); but we may readily integrate again, so as to obtain the relation between x and y. Thus, if $l'-l=\beta$, $y-l=\eta$,

$$\pm x = \sqrt{\frac{l^2 l'}{3\beta}} \log_e \left\{ \frac{2\beta}{\eta} - 1 + 2\sqrt{\frac{\beta^2}{\eta^2} - \frac{\beta}{\eta}} \right\}, \quad . \quad (J)$$

the constant being taken so that $x=0$ when $\eta=\beta$. This equation gives the height η at any point x in terms of one constant, viz. the maximum height of the wave. There is therefore (in a given canal) only one form of solitary wave of given maximum height. On either side the height diminishes without interruption, but does not (according to (J)) absolutely vanish at any finite distance. Accordingly there is no definite wave-length; but if we inquire what value of x corresponds to a given ratio of $\eta : \beta$, we get

$$x \propto \sqrt{\frac{l+\beta}{\beta}},$$

being greatest for the smallest waves.

Suppose, for example, that we regard the wave as ending where the height is one tenth of the maximum. Then

$$x : l = 2{\cdot}1\sqrt{1 + \frac{l}{\beta}}.$$

The shortest wave-length is when $\beta = l$; and then

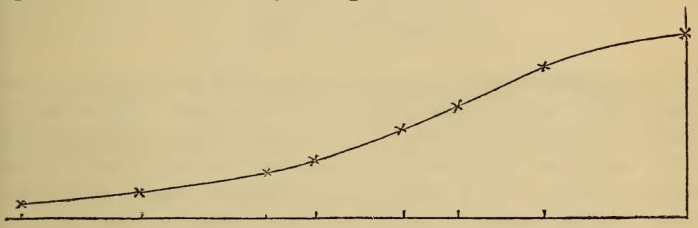
$$2x : l = 5{\cdot}96.$$

If $\beta = \frac{l}{3}$, $\quad 2x : l = 8{\cdot}4.$

If $\beta = \frac{l}{8}$, $\quad 2x : l = 12{\cdot}6.$

These results are in agreement with Russell's observations.

The form of the wave as determined by (J) is shown in the figure, half the wave only being drawn :—



The velocity of propagation is given by (H), which is Scott Russell's formula exactly. In words, the velocity of the wave is that due to half the greatest depth of the water.

Another of Russell's observations is now readily accounted for :—"It was always found that the wave broke when its elevation above the general level became equal, or nearly so, to the greatest depth. The application of mathematics to this circumstance is so difficult, that we confine ourselves to the mention of the observed fact"*. When the wave is treated as stationary, it is evident from dynamics that its height can never exceed that due to the velocity of the stream in the undisturbed parts; that is, $l' - l$ is less than $\frac{u_0^2}{2g}$. But $u_0^2 = gl'$, and therefore $l' - l$ is less than $\frac{1}{2}l'$, or $l' - l$ is less than l. When the wave is on the point of breaking, the water at the crest is moving with the velocity of the wave.

Periodic Waves in Deep Water.

The best known theory of these waves is that of Gerstner, Rankine, and Froude, in which the profile is trochoidal. The motion of each particle of the fluid is in a circle, which is described with uniform velocity. If h, κ be the coordinates of the centre of one of these circles, measured horizontally and downwards respectively, the position of the particle at time t

* Airy, 'Tides and Waves,' art. 401.

is given by

$$\left.\begin{aligned}\xi &= h + \mathrm{R}e^{-\frac{\kappa}{\mathrm{R}}}\sin\left(at + \frac{h}{\mathrm{R}}\right),\\ \eta &= \kappa + \mathrm{R}e^{-\frac{\kappa}{\mathrm{R}}}\cos\left(at + \frac{h}{\mathrm{R}}\right).\end{aligned}\right\}$$

It is not difficult to show that the motion represented by these equations satisfies the condition of continuity, and is consistent with the principles of fluid mechanics; but it involves a molecular rotation, whose amount is

$$ae^{-\frac{2\kappa}{\mathrm{R}}} \div 1 - e^{-\frac{2\kappa}{\mathrm{R}}}.$$

This molecular rotation, being constant for each particle, is not inconsistent with the properties of frictionless fluid when the motion is once set up; but it is known that a motion of this kind could not be generated from rest in such fluid by any natural force. We proceed to consider the theory of periodic waves in deep water when there is no molecular rotation.

As in the case of long waves, the problem may be reduced to one of steady motion by attributing to the water a velocity equal and opposite to that of the waves. If x be measured horizontally and y downwards from the surface, the conditions of continuity and of freedom from rotation are satisfied by

$$\left.\begin{aligned}\phi &= cx + \alpha e^{-\kappa y}\sin\kappa x,\\ \psi &= cy - \alpha e^{-\kappa y}\cos\kappa x;\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (\mathrm{A})$$

where ϕ and ψ are the equipotential and stream functions, c the velocity at a great depth, α a constant depending on the amplitude of the waves, and $\kappa = 2\pi \div \lambda$, λ being the wave-length or distance from crest to crest. The motion represented by (A) passes into a uniform horizontal flow at a great depth; and we have only to inquire how far the surface-condition of constant pressure can be satisfied.

If U be the resultant velocity at any point,

$$\begin{aligned}\mathrm{U}^2 &= \left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2 = c^2 + 2c\kappa\alpha e^{-\kappa y}\cos\kappa x + \kappa^2\alpha^2 e^{-2\kappa y}\\ &= c^2 + 2c\kappa(cy - \psi) + \kappa^2\alpha^2 e^{-2\kappa y},\end{aligned}$$

and therefore

$$\frac{p}{\rho} = \text{const.} + (g - \kappa c^2)y + c\kappa\psi - \tfrac{1}{2}\kappa^2\alpha^2 e^{-2\kappa y}. \quad . \quad . \quad (\mathrm{B})$$

Hence, when ψ is constant and α is so small that α^2 can be

neglected, p will also be constant, provided that

$$c=\sqrt{\frac{g}{\kappa}}=\sqrt{\frac{g\lambda}{2\pi}}. \quad . \quad . \quad . \quad . \quad . \quad (C)$$

If c has this value, the surface-condition is satisfied approximately, and (A) may be understood to represent a train of free periodic stationary waves, or, if the motion relatively to deep water be considered, a train of periodic waves advancing without change of type and with a uniform velocity c.

The profile of the wave is determined by the second of equations (A), in which ψ is made constant. By successive approximation we may deduce the value of y in terms of x. If ψ be taken so that the mean value of y is zero, we get

$$y=\frac{\alpha}{c}\left(1+\frac{5}{8}\frac{\kappa^2\alpha^2}{c^2}\right)\cos\kappa x-\frac{\kappa\alpha^2}{2c^2}\cos 2\kappa x+\frac{3}{8}\frac{\kappa^2\alpha^3}{c^3}\cos 3\kappa x,$$

which is correct as far as α^3. Let

$$\frac{\alpha}{c}\left(1+\frac{5}{8}\frac{\kappa^2\alpha^2}{c^2}\right)=a;$$

then

$$\left.\begin{aligned}\phi&=cx+ca\left(1-\frac{5}{8}\kappa^2a^2\right)e^{-\kappa y}\sin\kappa x,\\ \psi&=cy-ca\left(1-\frac{5}{8}\kappa^2a^2\right)e^{-\kappa y}\cos\ x;\end{aligned}\right\} \quad . \quad . \quad (D)$$

and for the equation of the surface,

$$y=a\cos\kappa x-\frac{\kappa a^2}{2}\cos 2\kappa x+\frac{3}{8}\kappa^2a^3\cos 3\kappa x. \quad . \quad . \quad (E)$$

From (B) we may obtain a closer approximation to the value of c. Expanding the exponential, we have (approximately)

$$\frac{p}{\rho}=\text{const.}+(g-\kappa c^2+\kappa^3\alpha^2)y+\ldots;$$

so that

$$c^2=\frac{g}{\kappa}+\kappa^2\alpha^2=\frac{g}{\kappa}+\kappa^2a^2c^2,$$

or

$$c^2=\frac{g}{\kappa}(1+\kappa^2a^2), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (F)$$

where $\kappa=2\pi\div\lambda$.

Formulæ (E) and (F) are given by Professor Stokes in a memoir published in the Cambridge Philosophical Transactions, vol. viii.

So long as the depth is everywhere sufficiently great in comparison with the length of the waves, uniformity of depth is

immaterial. For waves in water of constant finite depth l, the expression for ψ is

$$\psi = cy - \alpha \cos \kappa x \left\{ e^{-\kappa(y-l)} - e^{\kappa(y-l)} \right\},$$

and the velocity c is determined by

$$c^2 = \frac{g}{\kappa} \cdot \frac{e^{\kappa l} - e^{-\kappa l}}{e^{\kappa l} + e^{-\kappa l}}, \qquad \text{(G)}$$

which passes into (C) when l is considerable in comparison with λ. When l is small, we get from (G)

$$c^2 = \frac{g}{\kappa} \cdot \frac{2\kappa l}{2} = gl,$$

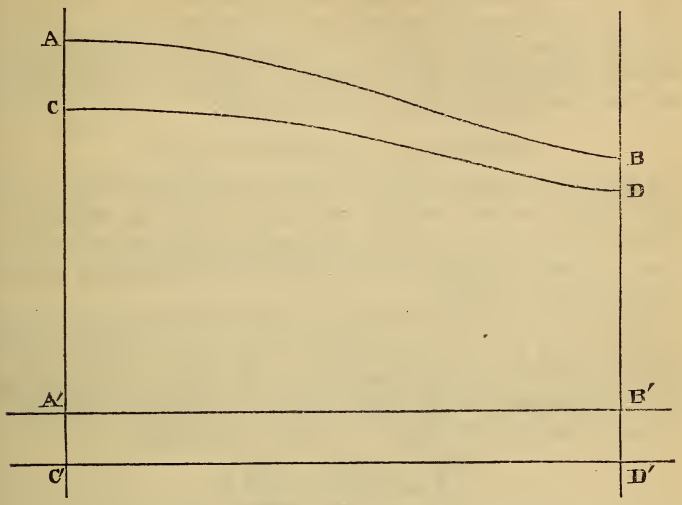
which is the formula proper for long waves. When obtained thus, it is applicable in the first instance only to waves of a particular type; but the fact that it is independent of κ or λ would lead us to the conclusion that the same formula would apply to a long wave of any type.

In one respect the theory of irrotational waves may be considered inferior to that of Rankine, which last is *exact*, in the sense that it is independent of any supposition as to the smallness of the waves. So far as I am aware, writers on this subject appear to think that it is only a question of mathematics to determine the form of irrotational waves of finite amplitude to any degree of approximation. But it seems to me by no means certain that any such type exists, capable of propagating itself unchanged with uniform velocity. I see no reason why the possibility of such waves in deep water should be taken for granted, when we know that in shallow water waves of finite height cannot be propagated without undergoing a gradual alteration of type.

One of the most interesting results of Professor Stokes's theory is the existence of a slow translation of the water near the surface in the direction of the wave. I propose to show that this superficial motion is an immediate consequence of the absence of molecular rotation, and that it is independent of the condition of constant pressure at the bounding surface.

Let A B be the surface from crest to hollow, and C D a neighbouring stream-line. Draw A′ B′, C′ D′, two stream-lines at such a depth that the steady motion of the fluid is uniform, and so as to include a total stream equal to that which flows between A B and C D. Then we have to show that a particle at A will take longer to reach B, than a particle at A′

takes to arrive at B′. Now if σ denotes the small breadth of the tube A D at any point, and v the velocity, the total stream



is σv and is constant. Denoting it by K, we have

$$v = \mathrm{K} \div \sigma.$$

The time t occupied by a particle in removing from A to B is therefore

$$t = \int \frac{ds}{v} = \int \frac{\sigma ds}{\mathrm{K}} = \text{area AD} \div \mathrm{K}.$$

And if t' represents the time between A′ and B′,

$$t' = \text{area A′D′} \div \mathrm{K},$$

K being the same in both cases, since the total streams are by supposition equal. Thus

$$t : t' = \text{area AD} : \text{area A′D′},$$

and it remains to prove that AD is greater than A′D′.

If we draw equipotential lines in such a manner that the small spaces cut off between them and AB, CD are squares, then we know that the same series of equipotential lines will divide the space between A′B′, C′D′, into small squares also. Now if a line be divided into a given number of parts, the sum of the squares of the parts will be a minimum when the parts are all equal. Hence the space AD is greater than if the squares described on the parts of AB were all equal, and therefore *à fortiori* greater than the space A′D′, which consists of

the sum of the squares of the same number of equal parts of a shorter line.

It follows that when a particle starting from A′ has arrived at B′, another particle starting at the same moment from A will fall short of B. Thus in a progressive wave the water near the surface has on the whole a motion of translation in the direction in which the waves advance.

Oscillations in Cylindrical Vessels.

If liquid contained in a cylindrical vessel of any section, whose generating lines are vertical and whose depth is uniform, be disturbed from the position of equilibrium, oscillations will ensue in consequence of the tendency of the fluid to recover its horizontal boundary.

Let us consider in the first place the small vibrations in two dimensions of a compressible fluid such as air when contained within a cylindrical rigid boundary. If x and y be the rectangular coordinates of any point, and ϕ the velocity potential, it is known that ϕ will satisfy over the whole area

$$\frac{d^2\phi}{dt^2}=a^2\left(\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}\right), \quad . \quad . \quad . \quad . \quad . \quad (A)$$

a being the velocity of sound; while round the contour

$$\frac{d\phi}{dn}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (B)$$

where $\frac{d\phi}{dn}$ denotes the rate of variation of ϕ in a normal direction.

Whatever the motion of the air may be, it can be analyzed into components of the harmonic type. Suppose that for one of these ϕ varies as $\cos \kappa at$; then, from (A),

$$\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}+\kappa^2\phi=0 \quad . \quad . \quad . \quad . \quad . \quad (C)$$

is an equation which ϕ must satisfy for the component vibration in question. The equations (C) and (B) can only be satisfied with certain definite values of κ; and the functions ϕ corresponding to these values are proportional to what may be called the *normal* functions of the air-system. We may denote these functions by u_κ. Any function arbitrary over the area can be expanded in a series of the functions u*.

Returning to the liquid-problem, we see that the elevation h of the surface at any point above the undisturbed position

* See on this subject several papers by the author, especially "General Theorems relating to Vibrations," Math. Society, Proceedings, vol. iv. No. 63, and Phil. Mag. December 1873.

may be expressed by the series

$$h=\Sigma\alpha_\kappa u_\kappa(xy), \quad \ldots\ldots\ldots \quad (D)$$

the quantities α being constants with respect to space, but dependent upon time. The potential energy of the displacement, *calculated on the hypothesis of a constant pressure on the surface*, will clearly be

$$\left.\begin{aligned} V&=g\rho\iint\cdot\int_0^h z\,dz\,.\,dx\,dy=\frac{g\rho}{2}\iint h^2dx\,dy \\ &=\frac{g\rho}{2}\iint(\Sigma\alpha_\kappa u_\kappa)^2dx\,dy=\frac{g\rho}{2}\Sigma\alpha_\kappa^2\iint u_\kappa^2dx\,dy \end{aligned}\right\}, \quad (E)$$

by the conjugate property of the functions u. This is the potential energy.

The motion of the fluid throughout the interior depends, according to a known theorem, only upon the motion of the surface; and the surface normal velocity

$$-\frac{d\phi}{dz}=\dot{h}=\Sigma\dot{\alpha}_\kappa u^\kappa.$$

If l be the depth, the complete value of ϕ is given by

$$-\phi=\Sigma\frac{e^{\kappa(z-l)}+e^{-\kappa(z-l)}}{\kappa(e^{-\kappa l}-e^{\kappa l})}\dot{\alpha}_\kappa u_\kappa(xy). \quad \ldots \quad (F)$$

For, in the first place, this value of ϕ satisfies Laplace's equation, inasmuch as each term u_κ satisfies the equation

$$\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\kappa^2u=0;$$

Secondly, (F) satisfies the condition imposed by the rigid cylindrical boundary, since $\frac{du_\kappa}{dn}=0$;

Thirdly, (F) makes $\frac{d\phi}{dz}=0$ when $z=l$;

And fourthly, when $z=0$, $-\frac{d\phi}{dz}=\dot{h}$.

The kinetic energy T may now be readily calculated:

$$\begin{aligned} T&=\frac{\rho}{2}\iiint\left\{\left(\frac{d\phi}{dx}\right)^2+\left(\frac{d\phi}{dy}\right)^2+\left(\frac{d\phi}{dz}\right)^2\right\}dx\,dy\,dz \\ &=\frac{\rho}{2}\iint\frac{d\phi}{dn}d\mathrm{S}, \end{aligned}$$

by Green's theorem, dS denoting an element of the surface bounding the mass, and $\frac{d\phi}{dn}$ the rate of variation of ϕ in a normal direction outwards. The surface S consists of three parts—the bottom of the vessel, the cylindrical side of the vessel, and the upper surface of the fluid. Over the first two of these, $\frac{d\phi}{dn}=0$, and thus

$$\mathrm{T}=-\frac{\rho}{2}\iint\left(\phi\frac{d\phi}{dz}\right)_{z=0}dx\,dy.$$

Now when $z=0$,

$$\phi=\Sigma\frac{e^{\kappa l}+e^{-\kappa l}}{e^{\kappa l}-e^{-\kappa l}}\frac{\dot{\alpha}_\kappa}{\kappa}u_\kappa,$$

$$\frac{d\phi}{dz}=-\Sigma\dot{\alpha}_\kappa u_\kappa,$$

so that

$$\mathrm{T}=\frac{\rho}{2}\Sigma\frac{\dot{\alpha}_\kappa^2}{\kappa}\frac{e^{\kappa l}+e^{-\kappa l}}{e^{\kappa l}-e^{-\kappa l}}\iint u_\kappa^2\,dx\,dy, \quad . \quad . \quad \text{(G)}$$

the product of any two functions u_κ, $u_{\kappa'}$ vanishing when integrated over the area.

We have now to calculate the work done by impressed forces corresponding to the displacement represented by $\delta\alpha_\kappa$. It must be remembered that these forces are limited to such as have a potential. Let δp denote the variable part of the pressure at the surface, supposed to remain in its position of rest, whether applied directly or due to impressed body-forces, then

$$\text{work done on system}=-\iint\delta p\,\delta h\,dx\,dy.$$

If δp be expanded in the series,

$$\delta p=\Sigma\beta_\kappa u_\kappa(xy),$$

$$\text{work}=-\iint\Sigma\beta_\kappa u_\kappa.\Sigma\delta\alpha_\kappa u_\kappa\,.\,dx\,dy$$

$$=-\Sigma\beta_\kappa\delta\alpha_\kappa\iint u_\kappa^2\,dx\,dy.$$

We can now form the equations of motion in terms of the generalized coordinates α_κ. By Lagrange's method,

$$\frac{\ddot{\alpha}_\kappa}{\kappa}\frac{e^{\kappa l}+e^{-\kappa l}}{e^{\kappa l}-e^{-\kappa l}}+g\alpha_\kappa=-\frac{\beta_\kappa}{\rho} \quad . \quad . \quad . \quad . \quad . \quad \text{(H)}$$

is the equation determining the variation of the coordinate α_κ, where

$$\beta_\kappa=\iint\delta p\,u_\kappa\,dx\,dy\div\iint u_\kappa^2\,dx\,dy.$$

When the oscillations are free, $\beta_\kappa=0$. If the period be τ_κ, and

the corresponding period for the air-vibration τ'_κ,

$$\left.\begin{aligned}\tau_\kappa &= 2\pi \div \sqrt{g\kappa \frac{e^{\kappa l}-e^{-\kappa l}}{e^{\kappa l}+e^{-\kappa l}}},\\ \tau'_\kappa &= 2\pi \div \kappa a.\end{aligned}\right\} \quad \ldots \quad (\mathrm{I})$$

If λ' be the wave-length of plane aerial vibrations having the period τ'_κ,

$$\kappa = 2\pi \div \lambda'.$$

If κl be very small, the ratio of periodic times is

$$\tau_\kappa : \tau'_\kappa = a : \sqrt{gl}, \quad \ldots \quad (\mathrm{J})$$

and is independent of κ. Hence the two problems of the vibrations of air and liquid are mathematically analogous whatever the initial circumstances may be; so that if the condensation in the first follows the same law initially as the elevation in the second, the correspondence will be preserved throughout the subsequent motion, if $a^2 = gl$. The initial circumstances, however, must be such as not to give prominence to the higher components, for which κl would no longer be small.

When κl is not negligible, we learn from formula (I) that the period increases with l until κl is moderately great, when it becomes sensibly

$$\tau_\kappa = 2\pi \div \sqrt{g\kappa}. \quad \ldots \quad (\mathrm{K})$$

In any case the period is independent of the density of the liquid.

Some careful observations on liquid vibrations have been recently made by Professor Guthrie*, with which it may be interesting to compare the results of theory. Professor Guthrie used troughs whose horizontal section was rectangular and circular. We will take the rectangular section first.

Confining ourselves to those modes of vibration which depend on only one horizontal coordinate, we may take for the normal functions

$$u = \cos \frac{n\pi x}{\mathrm{L}},$$

L being the length of the trough, n integral, and x being measured from one end. The corresponding value of κ is $\frac{n\pi}{\mathrm{L}}$. Hence, from (I), the length of the simple equivalent pendulum is

$$\frac{\mathrm{L}}{n\pi} \frac{e^{\frac{n\pi l}{\mathrm{L}}}+e^{-\frac{n\pi l}{\mathrm{L}}}}{e^{\frac{n\pi l}{\mathrm{L}}}-e^{-\frac{n\pi l}{\mathrm{L}}}}. \quad \ldots \quad (\mathrm{L})$$

* Phil. Mag. October and November 1875.

When $\frac{n\pi l}{\mathrm{L}}$ is considerable, (L) becomes

$$\frac{\mathrm{L}}{n\pi}, \quad \ldots \ldots \ldots \quad (\mathrm{M})$$

or, for a closer approximation,

$$\frac{\mathrm{L}}{n\pi}\left(1+2e^{-\frac{2n\pi l}{\mathrm{L}}}\right). \quad \ldots \ldots \quad (\mathrm{N})$$

Formula (M) was found by Professor Guthrie to agree with observation when $n=1$ or 2. The periods in the two cases are in the ratio $1:\sqrt{2}$, if the depth be sufficient.

If the depth bear a constant ratio to the length, (L) or (I) shows that the period is directly proportional to the square root of the linear dimension; and the same law will obtain when the depth is great, whatever the absolute value may be.

If $n=1$, the points of constant elevation occur when $x=\frac{\mathrm{L}}{2}$ (that is, in the middle of the length); and if $n=2$, when $x=\frac{\mathrm{L}}{4}$ or $\frac{3\mathrm{L}}{4}$. The maximum elevations (or depressions) are equal.

These results take into account inertia and gravity only. From some expressions in his paper Professor Guthrie would appear to attribute the effect of shallowness in increasing the period to friction. No doubt friction must act in this direction; but its immediate effect is on the amplitude, and not on the period. In all ordinary cases the action of insufficient depth may be sufficiently accounted for by the increase of the effective inertia due to the contraction of the channels along which the liquid flows, in the same way as the pitch of an organ-pipe is lowered by an obstruction at the mouth. In such vessels as those used by Professor Guthrie it may be doubted whether friction and capillarity have any sensible influence on the periodic time.

The theory for the circular trough depends on the class of functions named after Bessel, which are an extreme case of Laplace's spherical functions. For the symmetrical vibrations we have

$$u=\mathrm{J}_0(\kappa r), \quad \ldots \ldots \ldots \quad (\mathrm{O})$$

r being the radius vector; and if R be the radius of the vessel, κ is a root of

$$\mathrm{J}'_0(\kappa \mathrm{R})=0. \quad \ldots \ldots \quad (\mathrm{P})$$

If $x=\kappa\mathrm{R}$, the values of x satisfying (P) are 3·832, 7·015, 10·174, &c., of which only the first belongs to the cases experi-

mented on by Professor Guthrie. The approximate formula for the length of the simple equivalent pendulum corresponding to (N) is

$$\frac{R}{3{\cdot}832}(1+2e^{-7{\cdot}7\frac{l}{R}}), \quad \ldots \quad (Q)$$

or, when l is considerable,

$$R \div 3{\cdot}832 \text{ simply.} \quad \ldots \quad (R)$$

Professor Guthrie compares his observations with a pendulum of length $R \div 4$, and finds a fair agreement, which, however, would be improved by the substitution of the theoretical formula (R).

According to (O) the place of zero elevation and depression occurs when

$$r = \frac{2{\cdot}405}{3{\cdot}832} R = {\cdot}6277\, R.$$

According to observation,

$$r = \tfrac{2}{3} R = {\cdot}6667\, R.$$

From the Tables of Bessel's functions it appears that the amplitude at the edge of the vessel is ·403 of that at the centre. Professor Guthrie makes this ·5.

For the next set of vibrations in a circular dish u is of the form

$$u = \sin\theta\, J_1(\kappa r),$$

where the admissible values of κR are 1·841, 5·332, 8·536, &c. Hence for the gravest of this group the length of the equivalent pendulum is

$$R \div 1{\cdot}841. \quad \ldots \quad (S)$$

In this group of modes the elevation vanishes at all points along a certain diameter ($\theta = 0$).

In the third group we have

$$u = \sin 2\theta\, J_2(\kappa r),$$

and the admissible values of κR are 3·054, 6·705, 9·965, &c. For the gravest of these the length of the equivalent pendulum is

$$R \div 3{\cdot}054, \quad \ldots \quad (T)$$

if the depth be sufficient. The elevation vanishes along two perpendicular diameters ($\theta = 0$, $\theta = \frac{\pi}{2}$).

In the fourth group there would be three diameters for which $u = 0$; and the length of the pendulum isochronous with the gravest mode will be

$$R \div 4{\cdot}201. \quad \ldots \quad (U)$$

The frequencies of vibration in the three gravest modes, being inversely as the square root of the corresponding pendulum-lengths, are in the ratio

$$1 : 1{\cdot}29 : 1{\cdot}44.$$

Professor Guthrie's observations give for the value of these ratios

$$1 : 1{\cdot}31 : 1{\cdot}48.$$

Possibly too low a frequency is attributed to the gravest vibration from the effect of insufficient depth.

When the complete theory of the free vibrations of any system is thoroughly known, it is in general easy to investigate the effect of periodic forces. If u_1, u_2, &c. are the normal functions, and $\frac{2\pi}{n_1}$, $\frac{2\pi}{n_2}$, &c. the periods of the corresponding free vibrations, the effect of forces whose period is $\frac{2\pi}{p}$ can be expressed in terms of the effect produced by similar forces of infinite period, which last can be calculated statically. Thus, if the solution of the problem according to the equilibrium theory is

$$A_1 u_1 \cos pt + A_2 u_2 \cos pt + \ldots,$$

the true solution as modified by the inertia of the system will be

$$\frac{A_1 n_1^2}{n_1^2 - p^2} u_1 \cos pt + \frac{A_2 n_2^2}{n_2^2 - p^2} u_2 \cos pt + \ldots.$$

Let us calculate in this way the motion in a circular cylindrical basin due to a small horizontal force acting uniformly throughout the mass of liquid, but variable with the time according to the harmonic law. The equilibrium value of h (the elevation) is evidently

$$h = r \cos\theta \cos pt\,;$$

and the only difficulty consists in expressing r by a series of Bessel's functions J_1. It may be proved that

$$r = \frac{2J_1(\kappa r)}{(\kappa_1^2 - 1)J_1(\kappa_1)} + \frac{2J_1(\kappa_2 r)}{(\kappa_2^2 - 1)J_1(\kappa_2)} + \ldots, \quad . \quad (V)$$

where κ_1, κ_2, &c. are the roots of $J'_1(\kappa) = 0$, and the radius R is taken as unity. Thus the true value of h (after the motion has been going on long enough to be independent of initial circumstances) is

$$h = \frac{2n_1^2 \cos pt \cos\theta\, J_1(\kappa_1 r)}{(n_1^2 - p^2)(\kappa_1^2 - 1)J_1(\kappa_1)} + \ldots, \quad . \quad . \quad (W)$$

the summation being extended to all the admissible values of κ. The value of n^2 is given by

$$n^2 = g\kappa \frac{e^{\kappa l} - e^{-\kappa l}}{e^{\kappa l} + e^{-\kappa l}}.$$

If the system be at rest at $t=0$, and displaced according to the law $h = r\cos\theta$ (that is, with an inclined plane surface), the subsequent motion is given in rapidly converging series by

$$h = \frac{2J_1(\kappa_1 r)\cos\theta}{(\kappa_1^2 - 1)J_1(\kappa_1)}\cos n_1 t + \frac{2J_1(\kappa_2 r)\cos\theta}{(\kappa_2^2 - 1)J_1(\kappa_2)}\cos n_2 t + \ldots . \quad (X)$$

P.S.—Some recent observations on the periods of the oscillations of water in a large circular tank may be worth recording. The radius of the tank is 60·3 inches, and the depth about 43 inches. The oscillations were excited by dipping one or more buckets synchronously with the beats of a metronome set approximately beforehand. Soon after the withdrawal of the buckets the vibrations were counted (in most cases for five minutes), and the results reduced for a space of one minute.

Gravest symmetrical mode.—Frequency by observation 47·3. The theoretical result for an infinite depth is 47·32, and for actual depth 47·13.

Next highest symmetrical mode.—By observation, frequency = 64·1, by theory 64·02. In this case the correction for finite depth is insensible, and the length of the equivalent pendulum = R ÷ 7·015.

Gravest mode with one nodal diameter.—By observation, frequency = 30·0. By theory, for infinite depth 32·81, for actual depth 30·48.

One nodal diameter and one nodal circle.—By observation, frequency = 56·0, by theory 55·8. The length of equivalent pendulum = R ÷ 5·332.

Two nodal diameters.—By observation, frequency = 41·5. By theory, for infinite depth 42·09, for actual depth 41·59.

The agreement between theory and observation is as close as could be expected.

I have lately seen a memoir by M. Boussinesq (1871, *Comptes Rendus*, vol. lxxii.), in which is contained a theory of the solitary wave very similar to that of this paper. So far as our results are common, the credit of priority belongs of course to M. Boussinesq.